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Parametrised collision-free optimal motion planning algorithms in Euclidean spaces

Cesar A. Ipanaque Zapata [1] Jesús González [2]

Abstract

We describe parametrised motion planning algorithms for systems controlling objects represented by points that move without collisions in an even dimensional Euclidean space and in the presence of up to three obstacles with *a priori* unknown positions. Our algorithms are optimal in the sense that the parametrised local planners have minimal posible size.

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1 Introduction and main results

The design of explicit motion planners that are reasonably close to optimal is one of the challenges in modern robotics (see for instance Latombe [7] and LaValle [8]). As an answer to such a need, the concept of parametrised topological complexity has recently been introduced in [1] by Cohen, Farber and Weinberger in an attempt to increase the degree of universality and flexibility a motion planning has when performing under a variety of external conditions.

Let $p : E \to B$ be a fibration with path-connected fiber X. A *parametrised motion planning algorithm for* p is a function $\mathcal{A}$ assigning, to any pair of points $(e_1, e_2) \in E \times E$ with $p(e_1) = p(e_2)$, a continuous

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path $\gamma = \mathcal{A}(e_1, e_2)$ in E that starts at e_1, ends at e_2, and satisfies $p \circ \gamma = \overline{p(e_1)}$, where $\overline{b}$ stands for the constant path at $b \in B$. Mathematically, $\mathcal{A}$ is a (not necessarily continuous) section of the fibration $\gamma \mapsto (\gamma(0), \gamma(1))$ defined on the fibered path space

$$E_B^{[0,1]} = \{\gamma \in E^{[0,1]} : \ p \circ \gamma \text{ is a constant path}\}$$

and taking values in the fibered product

$$E \times_B E = \{(e_1, e_2) \in E \times E : \ p(e_1) = p(e_2)\},$$

where $E^{[0,1]}$ stands for the free-path space on E. In such a model, the space B is meant to parametrise all possible external conditions for a given system and, for any parameter $b \in B$, the fibre $p^{-1}(b)$ represents the corresponding space of actual states of the system where motion is to be planned.

For practical purposes, a parametrised motion planning algorithm should depend continuously on the pair of points $(e_1, e_2) \in E \times_B E$. Indeed, if the autonomous system performs within a noisy environment, then absence of continuity could lead to instability issues in the behavior of the parametrised motion planning algorithm. In other words, continuous parametrised motion planning algorithms are robust to noise. Unfortunately, a (global) continuous parametrised motion planning algorithm for p can exist only for a contractible fiber X (see [1, Proposition 4.5]). Yet, if X is not contractible, we could care about finding *local* continuous parametrised motion planning algorithms, i.e., parametrised motion planning algorithms s defined only on a certain open set of $E \times_B E$, to which we refer as the domain of definition of s. In these terms, a *parametrised motion planner for* p is a set of local continuous parametrised motion planning algorithms whose domains of definition cover $E \times_B E$. The *parametrised topological complexity of* p, $\mathrm{TC}_B(X)$, is then the minimal cardinality among parametrised motion planners for p, while a parametrised motion planner for p is said to be *optimal* if its cardinality is $\mathrm{TC}_B(X)$. Note that the reduced version of this invariant is presented in [1]. Because of our application minded goals, in this work we use the unreduced version. Summarizing, the components in the parametrised motion planning problem via topological complexity are:

1. The fibration $p : E \to B$ with fiber X. Here, a choice of a point $b \in B$ in the base space corresponds to a choice of the external conditions for the system.

2. Query pairs $e = (e_1, e_2) \in E \times_B E$. The point $e_1 \in E$ is the initial configuration of the query. The point $e_2 \in E$ is the goal configuration.

In the above setting, the goal is to either describe a parametrised motion planning algorithm, i.e., describe:

3. An open covering $U = \{U_1, \ldots, U_k\}$ of $E \times_B E$.
4. For each $i \in \{1, \ldots, k\}$, a parametrised motion planning algorithm, i.e., a continuous map $s_i \colon U_i \to E_B^{[0,1]}$ satisfying $s_i(e)\,(j) = e_{j+1}$ for any $e = (e_1, e_2) \in U_i$ and any $j \in \{0, 1\}$,

or, else, report that such system of sections does not exist.

Let X be a connected topological manifold of dimension at least 2. Consideration of the collision-free motion planning problem for n labelled robots, each with state space X, in the presence of m obstacles with *a priori* unknown positions, led Cohen, Farber and Weinberger to study the *Fadell-Neuwirth fibration* $\pi_{n+m,m} : F(X, n+m) \to F(X, m)$, given by

$$\pi_{n+m,m}(o_1, \ldots, o_m, x_1, \ldots, x_n) = (o_1, \ldots, o_m), \tag{1}$$

with fiber $F(X - \{\ m \text{ points}\}, n)$, where $F(Y, k)$ is the *ordered configuration space* of k distinct points on Y (see [4]). Explicitly,

$$F(Y, k) = \{(y_1, \ldots, y_k) \in Y^k \colon y_i \neq y_j \text{ for } i \neq j\}, \tag{2}$$

topologised as a subspace of the Cartesian power Y^k. In such a model, dynamics and other differential constraints are ignored, focusing primarily on the translations required to move the robots. In other words, robots and obstacles are represented by particles with infinitesimally small mass and volume, i.e., points in a Euclidean space $X = \mathbb{R}^d$. The position of the i-th robot is determined by $x_i \in \mathbb{R}^d$ in (1), while $o_j \in \mathbb{R}^d$ stands for the position of the j-th obstacle. In these terms, the condition $y_i \neq y_j$ in (2) reflects the collision-free and obstacle-avoidance requirements. Thus, a (local) parametrised motion planning algorithm for $\pi_{n+m,m}$ assigns to any pair of configurations (C_1, C_2) in (an open set of) $F(\mathbb{R}^d, n+m) \times_{F(\mathbb{R}^d,m)} F(\mathbb{R}^d, n+m)$ a continuous curve of configurations

$$\Gamma(t) \in F(\mathbb{R}^d, n+m), \quad t \in [0, 1],$$

such that $\pi_{n+m,m} \circ \Gamma = \overline{\pi_{n+m,m}(C_1)}$ and $\Gamma(i) = C_{i+1}$ for $i \in \{0,1\}$.

The parametrised topological complexity of $\pi_{n+m,m}$ when $X = \mathbb{R}^d$ has been computed by D. Cohen, M. Farber and S. Weinberger in [1] and [2]. The methods used therein are based on homotopy theory and, in particular, do not yield explicit motion planning algorithms. Inspired by our work in [10], we present an explicit parametrised motion planner for $\pi_{n+2,2}$ for any $d \geq 2$ even and $n \geq 1$. The planner has $2n+1$ regions of continuity and is optimal (in view of Theorem 2.6 below). The hypothesis that d be even is essential for this planner. In fact, the parametrised topological complexity is one unit larger when d is odd. On the other hand, the harder cases are those with d even, for then the calculation of the parametrised topological complexity in [2] is based on non-constructive techniques of obstruction theory.

In Section 2 we recall well-known results about the homotopy invariance of parametrised topological complexity. In particular, in Remark 2.5 we give explicit formulas describing how parametrised motion planners can be carried over from one space to another by means of a parametrised deformation. This allows us to construct in Section 3 the advertized parametrised motion planning algorithm for $\pi_{n+2,2}$ for any $d \geq 2$ even and $n \geq 1$. We emphasize that our algorithm works for $m = 2$, that is, for two obstacles. Indeed, the line determined by the pair of obstacles is key to our construction as it allows us to define desingularizations F^i in (5), sets $T_{i,j}$ in (7), deformations φ_i in (9), parametrised homotopies $\sigma_{i,j}$ in (10) and the algorithm $\overline{\Gamma}$ in (11).

The construction of optimal parametrised motion planners in the presence of more obstacles is far from being obvious and apparently calls for substantial adjustments. To better appreciate the complexity of the problem, in Section 4 we construct an optimal parametrised motion planner in the 2-D case of $\pi_{4,3}$, specifically, we describe an algorithm for motionplanning a single point-like robot moving in $\mathbb{R}^2$ so to avoid collisions with three fixed point-like obstacles whose positions in $\mathbb{R}^2$ are *a priori* unknown.

2 Preliminary results

After recalling from [1, 2] the basic properties of parametrised topological complexity, we give explicit formulas describing how parametrised motion planners can be carried over from one space to another by means of a parametrised deformation (Remark 2.5).

In the setting of the previous section, consider the evaluation fibration

$$\Pi : E_B^{[0,1]} \to E \times_B E, \quad \Pi(\gamma) = (\gamma(0), \gamma(1))\,. \tag{3}$$

A *parametrised motion planning algorithm* is, by definition, a section $s\colon E \times_B E \to E_B^{[0,1]}$ of the fibration Π, i.e., a (not necessarily continuous) map satisfying $\Pi \circ s = 1_{E\times_B E}$, where $1_{E\times_B E}$ denotes the identity map. When $E \times_B E$ has the homotopy type of a CW complex, a continuous parametrised motion planning algorithm for p exists if and only if the fiber X is contractible (see [1, Proposition 4.5]), which forces the following definition. The *parametrised topological complexity* $\mathrm{TC}_B(X)$ of a fibration $p : E \to B$ with fiber X is the Schwarz genus of the evaluation fibration (3). In other words the parametrised topological complexity of p is the smallest positive integer $\mathrm{TC}_B(X) = k$ for which the space $E \times_B E$ is covered by k open subsets $E \times_B E = U_1 \cup \cdots \cup U_k$ such that for any $i = 1, 2, \ldots, k$ there exists a continuous section $s_i : U_i \to E_B^{[0,1]}$ of Π over U_i (i.e., $\Pi \circ s_i = incl_{U_i}$, where $incl_{U_i}$ denotes the inclusion map). Thus, as noted in the introduction, we are using an unreduced notation for parametrised topological complexity.

Example 2.1. *Suppose that the fibers of $p : E \to B$ are convex sets. Given a pair of points $(e_1, e_2) \in E \times_B E$, we may move with constant velocity along the straight line segment connecting e_1 and e_2. This clearly produces a continuous parametrised algorithm for the parametrised motion planning problem for p. Thus we have $TC_B(X) = 1$.*

For the trivial fibration $E = B \times F \to B$, $\mathrm{TC}_B(X)$ coincides with Farber's topological complexity $\mathrm{TC}(X)$ of the fiber X, which is defined in terms of motion planning algorithms for a robot moving between initial-final configurations [5]. This means that trivial parametrisation does not add complexity (see [1, Example 4.2]).

The definition of $\mathrm{TC}_B(X)$ deals with open subsets of $E \times_B E$ admitting continuous sections of the evaluation fibration (3). Yet, for practical purposes, the construction of explicit parametrised motion planning algorithms is usually done by partitioning the whole space $E \times_B E$ into pieces, over each of which a continuous section for (3) is given. As discussed next, under mild conditions the resulting value of the parametrised topological complexity remains unaffected.

Recall that a topological space X is a *Euclidean Neighbourhood Retract* (ENR) if it can be embedded into an Euclidean space $\mathbb{R}^d$ with an open neighbourhood U, $X \subset U \subset \mathbb{R}^d$, admiting a retraction $r : U \to X$, $r\mid_U = 1_X$. In addition, a subspace $X \subset \mathbb{R}^d$ is an ENR if and only if it is locally compact and locally contractible, see [3, Chap. 4, Sect. 8]. This implies that finite-dimensional polyhedra, smooth manifolds and semi-algebraic sets are ENRs.

Definition 2.2. Let $E \times_B E$ be an ENR. A parametrised motion planning algorithm $s : E \times_B E \to E_B^{[0,1]}$ is said to be *tame* if $E \times_B E$ splits as a pairwise disjoint union $E \times_B E = F_1 \sqcup \cdots \sqcup F_k$, where each F_i is an ENR, and each restriction $s\mid_{F_i} : F_i \to E_B^{[0,1]}$ is continuous. The subsets F_i in such a decomposition are called *domains of continuity* for s.

Proposition 2.3. ([9, Proposition 2.2]) *For an ENR $E \times_B E$,* $\mathrm{TC}_B(X)$ *is the minimal number of domains of continuity $F_1, \ldots, F_k$ for tame parametrised motion planning algorithms $s : E \times_B E \to E_B^{[0,1]}$.*

A tame parametrised motion planning algorithm $s : E \times_B E \to E_B^{[0,1]}$ with continuity domains $F_1, \ldots, F_k$ yields an obvious motion-planning implementation. Namely, given a pair of initial-final configurations $(C_1, C_2) \in E \times_B E$, find the subset F_i such that $(C_1, C_2) \in F_i$ and take the path $s_i(C_1, C_2)$ as output.

Remark 2.4. We say that a tame parametrised motion planning algorithm $s : E \times_B E \to E_B^{[0,1]}$ is optimal when it admits $\mathrm{TC}_B(X)$ domains of continuity. At the end of the introduction we noted that the goal of this paper is the construction of optimal parametrised motion planners. We can now be more precise: we actually construct parametrised tame motion planning algorithms with the advertized optimality property.

The existence of a continuous parametrised motion planning algorithm on a subset U of $E \times_B E$ implies the existence of a corresponding continuous parametrised motion planning algorithm on any subset V of $E \times_B E$ deforming to U within $E \times_B E$ in the *parametrised context*. Such a fact is argued next in a constructive way, extending Example 6.4 in [6] to the parametrised case (the latter given for the non parametrised case). This of course suits best our implementation-oriented objectives.

Remark 2.5 (Constructing parametrised motion planning algorithms via parametrised deformations)**.** Let $s_U : U \to E_B^{[0,1]}$ be a continuous parametrised motion planning algorithm defined on a subset U of

$E \times_B E$. Suppose a subset $V \subseteq E \times_B E$ can be continuously deformed within $E \times E$ into V in the *parametrised context*, i.e., there is a homotopy $H : V \times [0,1] \to E \times E$ such that $H(v,0) = v$, $H(v,1) \in U$ and $h_1(v,-), h_2(v,-) \in E_B^{[0,1]}$ for any $v \in V$, where h_1, h_2 denote the Cartesian components of H, $H = (h_1, h_2)$. As schematized in the picture

$$\begin{tikzcd}
h_1(v,0) \arrow[d, dash] & h_2(v,0) \arrow[d, dash] \\
h_1(v,1) \arrow[r, dash] & h_2(v,1)
\end{tikzcd}$$

(where H runs from top to bottom and s_U runs from left to right), the path $s_U(H(v,1))$ in $E_B^{[0,1]}$ connects in sequence the points $h_i(v,1)$, $i \in \{1,2\}$, i.e.,

$$s_U(H(v,1))\,(i) = h_{i+1}(v,1), \quad i \in \{0,1\},$$

whereas the formula

$$s_V(v)(\tau) = \begin{cases} h_1(v, 3\tau), & 0 \leq \tau \leq \frac{1}{3}; \\ s_U(H(v,1))(3\tau - 1), & \frac{1}{3} \leq \tau \leq \frac{2}{3}; \\ h_2(v, 3 - 3\tau), & \frac{2}{3} \leq \tau \leq 1, \end{cases} .$$

defines a continuous section $s_V : V \to E_B^{[0,1]}$ of (3) over V. Summarizing: a parametrised deformation of V into U and a continuous parametrised motion planning algorithm defined on U determine an explicit continuous parametrised motion planning algorithm defined on V.

The final ingredient we need is the value of

$$\mathrm{TC}_{F(\mathbb{R}^d,m)}(F(\mathbb{R}^d - \{m \text{ points }\}, n)),$$

computed by Cohen-Farber-Weinberger in [1] and [2].

Theorem 2.6. ([1],[2]) *For any $m \geq 2$ and $n \geq 1$, the parametrised topological complexity of the problem of collision-free motion of n robots in the Euclidean d-space in the presence of m point obstacles with unknown a priori positions is given by*

$$\mathrm{TC}_{F(\mathbb{R}^d,m)}\Big(F(\mathbb{R}^d - \{m \textit{ points }\}, n)\Big) = \begin{cases} 2n + m, & \textit{if } d \geq 3 \textit{ is odd;} \\ 2n + m - 1, & \textit{if } d \geq 2 \textit{ is even.} \end{cases}$$

3 Parametrised motion planning algorithm for $\pi_{n+2,2}$ with $d \geq 2$ even

We present a parametrised motion planning algorithm for $\pi_{n+2,2}$ under the assumption (in force throughout this section) that $d \geq 2$ is even. The algorithm has $2n+1$ domains of continuity.

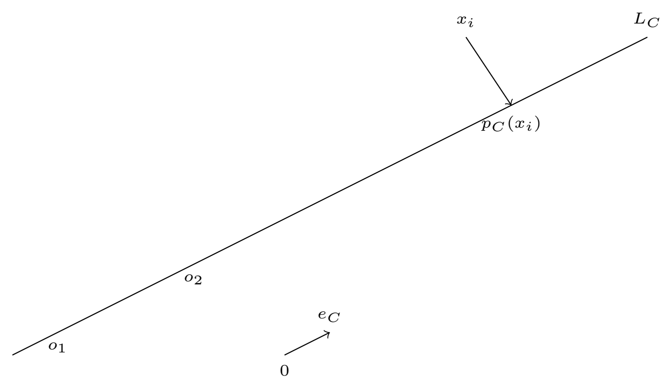


Figure 1: The line L_C, its orientation e_C, and the projection p_C.

For a configuration $C = (o_1, o_2, x_1, \ldots, x_n) \in F(\mathbb{R}^d, n+2)$, consider the affine line L_C through the points o_1 and o_2, oriented in the direction of the unit vector

$$e_C = \frac{o_2 - o_1}{\mid o_2 - o_1 \mid},$$

and let L'_C denote the line passing through the origin and parallel to L_C (with the same orientation as L_C). Let $p_C : \mathbb{R}^d \to L_C$ be the orthogonal projection, and let $\overline{\mathrm{cp}}(C)$ be the cardinality of the set $\{p_C(o_1), p_C(o_2), p_C(x_1), \ldots, p_C(x_n)\}$. Note that $\overline{\mathrm{cp}}(C)$ ranges from 2 to $n+2$. For $i \in \{2, \ldots, n+2\}$, let A_i denote the set of all configurations $C \in F(\mathbb{R}^d, n+2)$ with $\overline{\mathrm{cp}}(C) = i$. The various A_i are ENR's satisfying

$$\overline{A_i} \subset \bigcup_{j \leq i} A_j. \tag{4}$$

3.1 Desingularization

For a configuration $C = (o_1, o_2, x_1, \ldots, x_n) \in A_i$, set

$$\overline{\epsilon}(C) := \frac{1}{n+2} \min\{\mid p_C(x_r) - p_C(x_s) \mid : p_C(x_r) \neq p_C(x_s)\},$$

here $x_i = o_i$ for $i = 1, 2$. In addition, for C as above and $t \in [0,1]$, set

$$F^i(C,t) = \begin{cases} (o_1, o_2, \overline{z}_1(C,t), \dots, \overline{z}_n(C,t)), & \text{if } i < n+2; \\ C, & \text{if } i = n+2, \end{cases}$$

where $\overline{z}_j(C,t) = x_j + tj\overline{\epsilon}(C)e_C$ for $j = 1, \dots, n$. This defines a continuous "desingularization" deformation

$$F^i : A_i \times [0,1] \to F(\mathbb{R}^d, n+2) \tag{5}$$

of A_i into A_{n+2} inside $F(\mathbb{R}^d, n+2)$ (see Figure 2). Note that neither the lines L_C and L'_C nor their orientations change under the desingularization, i.e., $L_{F^i(C,t)} = L_C$, $L'_{F^i(C,t)} = L'_C$, and $e_{F^i(C,t)} = e_C$ for all $t \in [0,1]$. Indeed, we note that $\pi_{n+2,2}(F^i(C,t)) = \pi_{n+2,2}(C)$ for all $t \in [0,1]$.

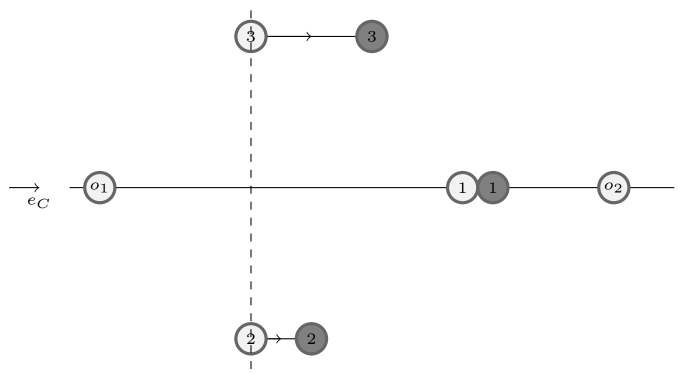


Figure 2: Desingularization.

3.2 The sets T_{ij}

We recall the sets A_{ij} and B_{ij} from [10]. For $i, j = 2, \dots, n+2$ let

$$A_{ij} := \{(C, C') \in A_i \times A_j \colon e_C \neq -e_{C'}\},$$
$$B_{ij} := \{(C, C') \in A_i \times A_j \colon e_C = -e_{C'}\}.$$

The sets A_{ij} and B_{ij} are ENR's (for they are semi-algebraic) covering $F(\mathbb{R}^d, n+2) \times F(\mathbb{R}^d, n+2)$ that satisfy

$$\overline{A_{ij}} \subseteq \bigcup_{r \leq i,\, s \leq j} A_{rs} \;\cup \bigcup_{r \leq i,\, s \leq j} B_{rs} \quad \text{and} \quad \overline{B_{ij}} \subseteq \bigcup_{r \leq i,\, s \leq j} B_{rs}, \tag{6}$$

in view of (4). Note that each B_{ij} does not intersect the subspace $F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$, i.e.,

$$B_{ij} \cap \Big(F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)\Big) = \varnothing,$$

because $(C, C') \in F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$ implies that $e_C = e_{C'}$ and thus $(C, C') \notin B_{ij}$. Consider subsets

$$T_{ij} := A_{ij} \cap \Big(F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)\Big). \tag{7}$$

The sets T_{ij} are ENR's (for they are semi-algebraic) covering $F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$ that satisfy

$$\overline{T_{ij}}_{\mathrm{rel}} \subseteq \bigcup_{r \leq i,\ s \leq j} T_{rs}, \tag{8}$$

in view of (6). Here, $\overline{T_{ij}}_{\mathrm{rel}}$ denotes the closure relative to the space $F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$, i.e.,

$$\overline{T_{ij}}_{\mathrm{rel}} = \overline{T_{ij}} \cap \Big(F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)\Big).$$

We also consider subset X of $W := F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$ defined by

$$X := \{(C, C') \in W\colon \text{ with both } C \text{ and } C' \text{ colinear}\}.$$

Here a configuration $C \in F(\mathbb{R}^d, n+2)$ is said to be colinear if in fact $C \in F(L_C, n+2)$.

Remark 3.1. The map $\varphi : A_{n+2} \times [0,1] \to F(\mathbb{R}^d, n+2)$ with coordinates $\varphi = (\varphi_1, \ldots, \varphi_{n+2})$ given by the formula

$$\varphi_i(C, t) = y_i + t(p_C(y_i) - y_i),\ i = 1, \ldots, n+2, \tag{9}$$

where $C = (o_1, o_2, x_1, \ldots, x_n) \in A_{n+2}$, $y_1 = o_1, y_2 = o_2$ and $y_{i+2} = x_i$ for each $i = 1, \ldots, n$, defines a continuous deformation of A_{n+2} onto $F(L_C, n+2)$ inside $F(\mathbb{R}^d, n+2)$ depicted in Figure 3. Note that, $\varphi_i(C, t) = o_i$ for any $t \in [0,1]$ and each $i = 1, 2$.

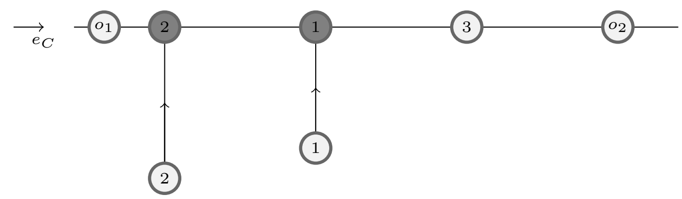


Figure 3: Deformation of A_{n+2} onto $F(L_C, n+2)$ inside $F(\mathbb{R}^d, n+2)$.

3.3 Deformations σ_{ij}

Next we define parametrised homotopies

$$(10) \qquad \sigma_{ij} : T_{ij} \times [0,1] \to F(\mathbb{R}^d, n+2) \times F(\mathbb{R}^d, n+2),$$

deforming T_{ij} into X, i.e., such that

1. $\sigma_{ij}((C, C'), 0) = (C, C')$ and $\sigma_{ij}((C, C'), 1) \in X$.
2. $\pi_{n+2,2} \circ \sigma^1_{ij}((C, C'), -) = \overline{\pi_{n+2,2}(C)}$ and $\pi_{n+2,2} \circ \sigma^2_{ij}((C, C'), -) = \overline{\pi_{n+2,2}(C')}$, where $\sigma^1_{ij}, \sigma^2_{ij}$ denote the Cartesian components of σ_{ij}, i.e., $\sigma_{ij} = (\sigma^1_{ij}, \sigma^2_{ij})$. Recall that $\pi_{n+2,2}(C) = \pi_{n+2,2}(C')$ for any $(C, C') \in T_{ij}$.

The deformation σ_{ij}: Given a pair $(C, C') \in T_{ij}$, we first apply the desingularization deformations $F^i(C, t)$ and $F^j(C', t)$ in order to take the pair (C, C') into a pair of configurations $(C_1, C'_1) \in T_{n+2,n+2}$ (recall $\pi_{n+2,2}(C_1) = \pi_{n+2,2}(C) = \pi_{n+2,2}(C') = \pi_{n+2,2}(C'_1)$). Next we apply the linear deformation (9), in order to take the pair (C_1, C'_1) into a pair of colinear configurations $(C_2, C'_2) \in X$. The deformation σ_{ij} is the concatenation of the two deformations just described.

3.4 Section over X

Recall that $X \subset F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$ is the set of pairs (C, C') of colinear configurations. Note that, $L_C = L_{C'} =: L_{C,C'}$. We construct a continuous parametrised motion planning algorithm

$$(11) \qquad \overline{\Gamma} \colon X \to F(\mathbb{R}^d, n+2)^{[0,1]}_{F(\mathbb{R}^d,2)}$$

provided d is even (this is the only place where the hypothesis about the parity of d is used).

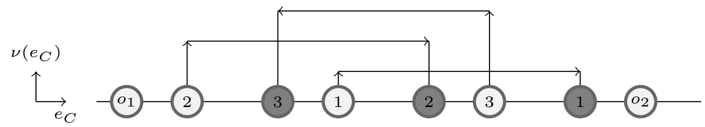


Figure 4: Section over X. Vertical arrows pointing upwards (downwards) describe the first (last) third of the path $\overline{\Gamma}^{C,C'}$, whereas horizontal arrows describe the middle third of $\overline{\Gamma}^{C,C'}$.

Let ν be a fixed unitary tangent vector field on S^{d-1}, say

$$\nu(x_1, y_1, \ldots, x_\ell, y_\ell) = (-y_1, x_1, \ldots, -y_\ell, x_\ell)$$

with $d = 2\ell$. Given two configurations $C = (o_1, o_2, x_1, \ldots, x_n)$ and $C' = (o_1, o_2, x'_1, \ldots, x'_n)$ in $F(L_{C,C'}, n+2)$, let $\overline{\Gamma}^{C,C'}$ be the path in the fiber $\pi^{-1}_{n+2,2}(o_1, o_2) \subset F(\mathbb{R}^d, n+2)$ from C to C' depicted in Figure 4. Explicitly, if $C = (o_1, o_2, x_1, \ldots, x_n)$ and $C' = (o_1, o_2, x'_1, \ldots, x'_n)$, then the path $\overline{\Gamma}(C, C')$ in the fiber $\pi^{-1}_{n+2,2}(o_1, o_2) \subset F(\mathbb{R}^d, n+2)$ from C to C' has components $(o_1, o_2, \overline{\Gamma}^{C,C'}_1, \ldots, \overline{\Gamma}^{C,C'}_n)$ defined by

$$\overline{\Gamma}^{C,C'}_i(t) = \begin{cases} x_i + (3ti)v(e_C), & \text{for } 0 \leq t \leq \frac{1}{3}; \\ x_i + iv(e_C) + (3t-1)(x'_i - x_i), & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ x'_i + i(3-3t)v(e_C), & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

3.5 Repacking regions of continuity

As explained in Remark 2.5, we can combine the continuous parametrised motion planning algorithm $\overline{\Gamma}$ with the concatenation of the parametrised deformations discussed so far to obtain continuous parametrised motion planning algorithms

$$T_{i,j} \to F(\mathbb{R}^d, n+2)^{[0,1]}_{F(\mathbb{R}^d,2)}, \tag{12}$$

for $i, j = 2, \ldots, n+2$. The corresponding upper bound

$$\mathrm{TC}_{F(\mathbb{R}^d,2)}\Big(F(\mathbb{R}^d - \{\text{ 2 points }\}, n)\Big) \leq (n+1)^2$$

is improved by repacking these regions of continuity. Set

$$W_\ell = \bigcup_{i+j=\ell} T_{ij}$$

for $\ell = 4, \ldots, 2n+4$. In view of (8), the sets assembling each W_ℓ are topologically disjoint in the sense that $\overline{T_{ij}}_{\mathrm{rel}} \cap T_{i'j'} = \varnothing$, provided $i+j = i'+j'$ and $(i,j) \neq (i',j')$, so the sets W_ℓ are ENR's covering $F(\mathbb{R}^d, n+2) \times_{F(\mathbb{R}^d,2)} F(\mathbb{R}^d, n+2)$ on each of which the corresponding algorithms in (12) assemble a continuous parametrised motion planning algorithm. We have thus constructed a tame parametrised motion planning algorithm for $\pi_{n+2,2} \colon F(\mathbb{R}^d, n+2) \to F(\mathbb{R}^d, 2)$ having $2n+1$ regions of continuity $W_4, W_5, \ldots, W_{2n+4}$ (see Figure 5).

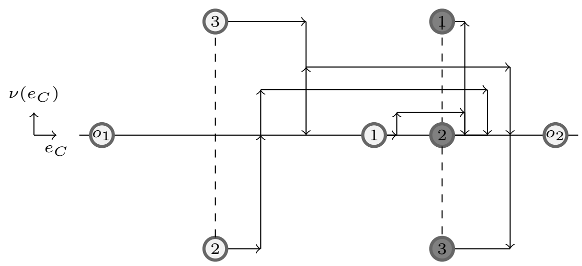


Figure 5: The motion planning algorithm for $\pi_{n+2,2}$.

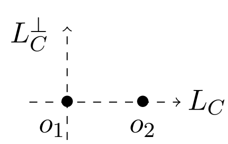
4 Parametrised motion planning algorithm for $\pi_{4,3}$ with $d = 2$

We present a parametrised motion planning algorithm for $\pi_{4,3}$ in the 2D case. The algorithm has four domains of continuity, so its optimality follows from Theorem 2.6. As in Section 3, we consider a unitary tangent vector field ν on S^1, say the one given by $\nu(x_1, y_1) = (-y_1, x_1)$.

For a configuration $C = (o_1, o_2, o_3, x) \in F(\mathbb{R}^2, 4)$, consider the affine line L_C through the points o_1 and o_2 oriented in the direction of the unit vector

$$e_C = \frac{o_2 - o_1}{\mid o_2 - o_1 \mid},$$

and let $L_C^\perp$ denote the affine line perpendicular to L_C that passes through the point o_1 and is oriented in the direction of the unit vector $\nu(e_C)$.



Let $p_C : \mathbb{R}^2 \to L_C$ and $p_C^{\perp} : \mathbb{R}^2 \to L_C^{\perp}$ be the orthogonal projections, and set

$$\begin{aligned}
\mathrm{cp}_o^{\perp}(C) &= \mid \{p_C^{\perp}(o_1), p_C^{\perp}(o_2), p_C^{\perp}(o_3)\} \mid, \\
\mathrm{cp}^{\perp}(C) &= \mid \{p_C^{\perp}(o_1), p_C^{\perp}(o_2), p_C^{\perp}(o_3), p_C^{\perp}(x)\} \mid, \\
\mathrm{cp}_o(C) &= \mid \{p_C(o_1), p_C(o_2), p_C(o_3)\} \mid, \\
\mathrm{cp}(C) &= \mid \{p_C(o_1), p_C(o_2), p_C(o_3), p_C(x)\} \mid,
\end{aligned}$$

where $\mid S \mid$ denotes the cardinality of the set S. Note that

$$\mathrm{cp}_o^{\perp}(C) \in \{1,2\},\ \mathrm{cp}^{\perp}(C) \in \{1,2,3\},\ \mathrm{cp}_o(C) \in \{2,3\},\ \mathrm{cp}(C) \in \{2,3,4\},$$

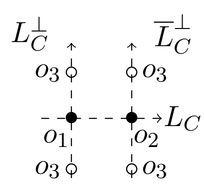
although not all combinations are achievable for a point (C, C') in the fibered product $F(\mathbb{R}^2,4)\times_{F(\mathbb{R}^2,3)}F(\mathbb{R}^2,4)$. To be precise, for $i, j \in \{1,2\}$, $r, s \in \{2,3\}$ and $k, l \in \{3,4\}$, consider the subsets $T^1_{i,j}$, $T^{2,2}_{r,s}$ and $T^{2,3}_{k,l}$ of $F(\mathbb{R}^2,4) \times_{F(\mathbb{R}^2,3)} F(\mathbb{R}^2,4)$ consisting of the pairs (C, C') satisfying the following list of conditions:

In $T^1_{i,j}$: $\mathrm{cp}_o^{\perp}(C) = 1,\ \mathrm{cp}^{\perp}(C) = i$ and $\mathrm{cp}^{\perp}(C') = j$.

In $T^{2,2}_{r,s}$: $\mathrm{cp}_o^{\perp}(C) = 2,\ \mathrm{cp}_o(C) = 2,\ \mathrm{cp}(C) = r$ and $\mathrm{cp}(C') = s$.

In $T^{2,3}_{k,l}$: $\mathrm{cp}_o^{\perp}(C) = 2,\ \mathrm{cp}_o(C) = 3,\ \mathrm{cp}(C) = k$ and $\mathrm{cp}(C') = l$.

Thus, for $(C, C') \in T^1_{i,j}$, the three common obstacles in C and C' lie in L_C, whereas the non-obstacle in C (respectively C') lies in L_C if and only if $i = 1$ (respectively, $j = 1$). Likewise, for $(C, C') \in T^{2,2}_{r,s}$, the four relative positions of the three common obstacles in C and C' can be depicted as



whereas the non-obstacle in C (C', respectively) lies on

$$L_C^{\perp} \cup \overline{L}_C^{\perp}$$

if and only if $r = 2$ ($s = 2$, respectively). Lastly, for $(C, C') \in T^{2,3}_{k,l}$, the third common obstacle o_3 lies outside

$$L_C \cup L_C^{\perp} \cup \overline{L}_C^{\perp},$$

while the non-obstacle in C (C', respectively) determines a fourth projection on L_C if and only if $k = 4$ ($l = 4$, respectively). We thus have:

Corollary 4.1. *The various ENR's $T^1_{i,j}$, $T^{2,2}_{r,s}$ and $T^{2,3}_{k,l}$ give a partition of the fibered product $F(\mathbb{R}^2, 4) \times_{F(\mathbb{R}^2,3)} F(\mathbb{R}^2, 4)$.*

Continuous parametrized motion planning algorithms on the various $T^{*,*}_{*,*}$ are described next. In each case, motion is meant to be performed at constant speed along the indicated path. The following conventions are in force in the next pictures: *(i)* The obstacle o_3 is sometimes omitted when it lies in L_C and is not relevant. *(ii)* The auxiliary dashed oriented lines L_C and $L^\perp_C$ are drawn without specifying their names. *(iii)* The positive (negative, respectively) hemiplane $H_{C,+}$ ($H_{C,-}$, respectively) determined by L_C is the one located in the positive (negative, respectively) $L^\perp_C$-direction, likewise we have positive and negative hemiplanes $H^\perp_{C,+}$ and $H^\perp_{C,-}$ determined by $L^\perp_C$, where signs are determined by the L_C-direction. *(iv)* We set $C = (o_1, o_2, o_3, x)$ and $C' = (o_1, o_2, o_3, x')$, and let $d(u, v)$ stand for the Euclidean distance between the points $u, v \in \mathbb{R}^2$.

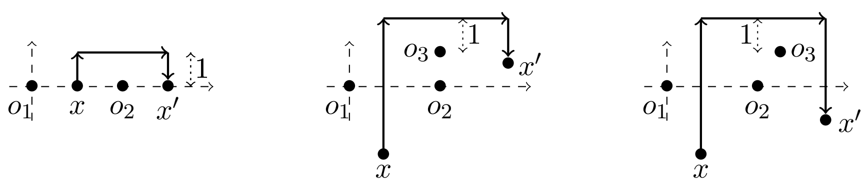


Figure 6: $T^1_{1,1}$ (left), $T^{2,2}_{3,3}$ (center) and $T^{2,3}_{4,4}$ (right)

Parametrised motion planning in $T^1_{1,1}$, $T^{2,2}_{3,3}$ and $T^{2,3}_{4,4}$ uses the paths in $H_{C,+}$ depicted in Figure 6 and constructed in terms of three lines, namely, the $L^\perp_C$-parallel lines through x and x', and the L_C-parallel line having p_C projection $1 + \max\{p_C(o_1), p_C(o_2), p_C(o_3)\}$.

As depicted in Figure 7, parametrised motion planning in $T^1_{1,2}$ and $T^1_{2,1}$ uses straight lines. Parametrised motion planning in $T^1_{2,2}$ and $T^{2,2}_{2,2}$ uses the paths depicted in Figure 8 and constructed in terms of the L_C-parallel lines through x and x' and the $L^\perp_C$-parallel line in $H^\perp_{C,+}$ whose distance to o_1 is $\delta_1 = \frac{1}{2}\min\{d(o_1, o_2), d(o_1, o_3)\}$, in the case of $T^1_{2,2}$, and $\delta_2 = \frac{1}{2}d(o_1, o_2)$, in the case of $T^{2,2}_{2,2}$.

Parametrised motion planning in $T^{2,2}_{2,3}$ and $T^{2,2}_{3,2}$ uses the paths depicted in Figure 9 and constructed in terms of four lines $\ell_1, \ldots, \ell_4$. For

Figure 7: $T^1_{1,2}$ (left) and $T^1_{2,1}$ (right)

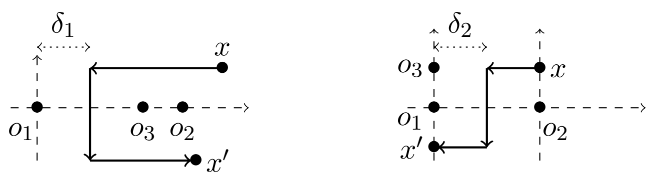


Figure 8: $T^1_{2,2}$ (left) and $T^{2,2}_{2,2}$ (right)

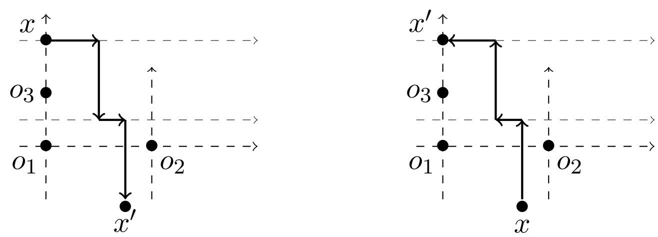


Figure 9: Parametrised motion planning in $T^{2,2}_{2,3}$ (left) and $T^{2,2}_{3,2}$ (right)

instance, in the case of $T^{2,2}_{2,3}$, ℓ_1 is the L_C-parallel line through x; ℓ_2 is the $L_C^{\perp}$-parallel line through the middle point between o_1 and o_2; ℓ_3 is the L_C-parallel line through the middle point between o_3 and the obstacle o_i having $i \in \{1,2\}$ and $p_C(o_3) = p_C(o_i)$; ℓ_4 is the $L_C^{\perp}$-parallel line through x'.

Parametrised motion planning in $T^{2,3}_{3,3}$ is best pictured in terms of the grid depicted in Figure 10, where vertical (horizontal, respectively) dashed lines represent the three (two, respectively) different values in $\{p_C(o_1), p_C(o_2), p_C(o_3)\}$ ($\{p_C^{\perp}(o_1), p_C^{\perp}(o_2), p_C^{\perp}(o_3)\}$, respectively) determined by an element in $T^{2,3}_{3,3}$. Solid lines are then constructed to be right in between two consecutive dashed lines, except for the right-most vertical solid line that is chosen to be one unit to the right of the right-most dashed vertical line. In such a setting, obstacles are located at the intersections of dashed lines (there are only six possibilities), whereas x and x' are located along vertical dashed lines. Parametrised motion planning from x to x' then uses the simple path constructed in terms of the three solid lines in Figure 10 together with the L_C-parallel lines connecting x and x' to the first solid vertical line on their right.

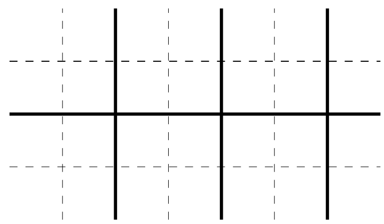


Figure 10: Grid for $T^{2,3}_{3,3}$

Parametrised motion planning in $T^{2,3}_{3,4}$ and $T^{2,3}_{4,3}$ uses the strategy in the previous paragraph, with a single modification. Namely, in the case of $T^{2,3}_{3,4}$ ($T^{2,3}_{4,3}$, respectively), so that x' (x, respectively) does not lie on some of the vertical dashed lines of Figure 10, the corresponding L_C-parallel line through x' (x, respectively) is replaced by the $L_C^{\perp}$-parallel line connecting x' (x, respectively) to the solid lines in Figure 10.

The discussion above is still not enough to get the desired optimal parametrised motion planner with 4 domains. We need a suitable repacking of the various T's. Explicitly, we consider the partition of $F(\mathbb{R}^2,4)\times_{F(\mathbb{R}^2,3)} F(\mathbb{R}^2,4)$ given by the ENR's

$$
\begin{aligned}
W_1 &= T^1_{1,1}\cup T^{2,2}_{3,3}\cup T^{2,3}_{4,4}, \\
W_2 &= T^1_{2,2}\cup T^{2,2}_{2,2}, && (13)\\
W_3 &= T^1_{1,2}\cup T^1_{2,1}\cup T^{2,2}_{2,3}\cup T^{2,2}_{3,2}\cup T^{2,3}_{3,3} \quad \text{and} && (14)\\
W_4 &= T^{2,3}_{3,4}\cup T^{2,3}_{4,3}. && (15)
\end{aligned}
$$

Proposition 4.2. *The parametrised motion planning algorithms on the various T's assemble a parametrised motion planner with domains of definition W_i for $1\le i\le 4$.*

Proof. Continuity of the parametrised motion planning algorithm for W_1 follows by direct inspection of Figure 6. We prove continuity in the other three cases by observing that the unions in (13)–(15) are topological. In W_2 we have

$$\overline{T^1_{2,2}}\cap T^{2,2}_{2,2}=\varnothing$$

because the condition $\mathrm{cp}_o^{\perp}(C) = 1$ defining $T^1_{2,2}$, which is inherited by $\overline{T^1_{2,2}}$, is incompatible with the defining condition $\mathrm{cp}_o^{\perp}(C) = 2$ in $T^{2,2}_{2,2}$. Likewise, the equality

$$T^1_{2,2} \cap \overline{T^{2,2}_{2,2}} = \varnothing$$

holds since the condition $\mathrm{cp}_o(C) = 2$ defining $T^{2,2}_{2,2}$, which is inherited by $\overline{T^{2,2}_{2,2}}$, is incompatible with the condition $\mathrm{cp}_o(C) = 3$ forced in $T^1_{2,2}$. On the other hand, the equalities

$$\overline{T^{2,3}_{3,4}} \cap T^{2,3}_{4,3} = \varnothing = T^{2,3}_{3,4} \cap \overline{T^{2,3}_{4,3}}$$

in W_4 follow by looking at conditions $\mathrm{cp}(C)$ and $\mathrm{cp}(C')$, respectively. Finally, the fact that the first four T-pieces of W_3 are topologically separated from the rest of the pieces comes by looking at:

- $\mathrm{cp}^{\perp}(C)$ for the $T^1_{1,2}$ piece;
- $\mathrm{cp}^{\perp}(C')$ for the $T^1_{2,1}$ piece;
- $\mathrm{cp}(C)$ for the $T^{2,2}_{2,3}$ piece;
- $\mathrm{cp}(C')$ for the $T^{2,2}_{3,2}$ piece.

The topologically-separated condition for the last piece $T^{2,3}_{3,3}$ of W_3 is a bit more elaborated:

- $\overline{T^{2,3}_{3,3}} \cap T^1_{1,2} = \varnothing$ because of the respective conditions on $\mathrm{cp}(C)$;
- $\overline{T^{2,3}_{3,3}} \cap T^1_{2,1} = \varnothing$ because of the respective conditions on $\mathrm{cp}(C')$;
- $\overline{T^{2,3}_{3,3}} \cap T^{2,2}_{2,3} = \varnothing$ as $p_C(x') \in \{p_C(o_1), p_C(o_2), p_C(o_3)\}$ holds in $\overline{T^{2,3}_{3,3}}$ but not in $T^{2,2}_{2,3}$;
- $\overline{T^{2,3}_{3,3}} \cap T^{2,2}_{3,2} = \varnothing$ as $p_C(x) \in \{p_C(o_1), p_C(o_2), p_C(o_3)\}$ holds in $\overline{T^{2,3}_{3,3}}$ but not in $T^{2,2}_{3,2}$. $\square$

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Cesar A. Ipanaque Zapata
Departamento de Matemática, Universidade de São Paulo,
Instituto de Matemática e Estadística -IME/USP,
R. do Matão, 1010 - Butantã,
CEP: 05508-090 - São Paulo, Brasil,
cesarzapata@usp.br

Jesús González
Departamento de Matemáticas,
Centro de Investigación y de Estudios Avanzados del I.P.N.,
Av. Instituto Politécnico Nacional número 2508,
San Pedro Zacatenco, México City 07000, México,
jesus@math.cinvestav.mx

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