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De Rham currents and the Chevalley-Eilenberg resolution*

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Abstract

We show how Serre's interpretation of the algebras of distributions on a formal group can be used to construct the Chevalley-Eilenberg resolution for a Lie algebra. This construction does not require associativity; when applied to non-associative products, it produces invariant differential forms on homogeneous spaces.

1 Introduction

The purpose of this, essentially, expository note is to show how to define the Chevalley-Eilenberg resolution for a Lie algebra $\mathfrak{g}$ without using the Jacobi identity in $\mathfrak{g}$ (or associativity in the universal enveloping algebra of $\mathfrak{g}$). The approach that we take here is to consider the cohomology of Lie algebras as an "infinitesimal cohomology of Lie groups". The idea behind it comes from Serre [1, 6] who showed that a Lie algebra can be obtained from a Lie group in two steps. First one passes from a Lie group G to the Hopf algebra $D(G)$ of Schwartz distributions on G supported at the unit of G. The Hopf algebra $\mathcal{D}(G)$ turns out to be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$; in particular, $\mathfrak{g}$ is the Lie algebra of the primitive elements in $\mathcal{D}(G)$. We have two functors

$$\begin{array}{ccccc} \text{Lie groups} & \to & \text{Hopf algebras} & \to & \text{Lie algebras} \\ G & \mapsto & \mathcal{D}(G) = U(\mathfrak{g}) & \mapsto & \mathfrak{g} \end{array}$$

*__Invited paper.__

whose composition assigns to a Lie group G its tangent Lie algebra $\mathfrak{g}$. One may also think of this construction as producing the universal enveloping algebra of a Lie algebra in terms of the product in the corresponding Lie group. At no point it uses the associativity of G, so it easily translates into the context of non-associative algebra, see [3, 4].

Serre's approach suggests that the Hopf algebra of distributions $\mathcal{D}(G)$ may be a more fundamental infinitesimal object than the Lie algebra $\mathfrak{g}$. It turns out that replacing the distributions on G by more general *de Rham currents* on G (that is, continuous linear functions on differential forms, rather than just on functions), one obtains the Chevalley-Eilenberg resolution of the Lie algebra of G which is used in the definition of the Lie algebra cohomology. This definition can be stated in a way that does not even mention Lie algebras.

The Chevalley-Eilenberg resolution obtained in this fashion is naturally a cocommutative differential graded Hopf algebra. It can also be thought of as the universal enveloping algebra of a certain differential graded Lie algebra, namely the *cone* of the Lie algebra of G. This is not surprising since currents are a differential graded version of distributions.

In differential geometry one encounters unital, non-associative[1] and, possibly, locally defined, products. They arise naturally on homogeneous spaces; Loos [2] shows how the theory of symmetric spaces may be developed in the framework of the non-associative algebra. The construction described here produces a differential graded Hopf algebra similar to the Chevalley-Eilenberg resolution for any non-associative local product. For such products, the standard complex calculating the cohomology of a Lie algebra is replaced by the complex of invariant forms on the corresponding homogeneous space.

The approach we take in this note should be known to experts but, for some reason, is hard to find in the literature. This note is based on a talk given by the author at the LieJor Online Seminar: Algebras, Representations, and Applications on September 3, 2020; this accounts for the informal style of the exposition. We assume the knowledge of basic facts about Lie algebras, Hopf algebras and chain complexes.

[1] meaning "not necessarily associative".

2 Distributions and universal enveloping algebras

2.1 Distributions

Recall that a *Schwartz distribution* on $\mathbb{R}^n$ is a continuous linear function on some given space of smooth functions (called *test functions*) on $\mathbb{R}^n$. A wide class of distributions is determined by compactly supported continuous functions: for a fixed compactly supported function $\rho : \mathbb{R}^n \to \mathbb{R}$ the distribution T_ρ is defined as

$$T_\rho(f) = \int_{\mathbb{R}^n} \rho f.$$

The partial derivatives of a distribution T are defined by

$$\frac{\partial T}{\partial x_i}(f) = T\left(\frac{\partial f}{\partial x_i}\right),$$

and the directional derivatives are defined in the same way. They are defined whether or not the distribution is defined by a compactly supported function and only require the differentiability of the test functions.

It is said that the distribution T is supported on a set $A \in \mathbb{R}^n$ if $T(f) = 0$ for any function f such that $f|_A = 0$. There is a class of distributions whose support is a single point. The so-called *Dirac's delta*, defined by

$$\delta(f) = f(0),$$

has support in the origin in $\mathbb{R}^n$. The partial derivatives of the Dirac's delta are also supported at the origin and so are all their linear combinations. We denote the vector space spanned by the Dirac's delta and its derivatives by $\mathcal{D}_0$ and refer to it as the space of distributions supported at 0. The space $\mathcal{D}_0$ is graded by the order of the derivatives: δ has degree 0, its first derivatives are of degree 1, and so on. We will use the following notation for the partial derivatives of the Dirac's delta:

$$\xi_1^{d_1} \ldots \xi_n^{d_n} = \frac{\partial^{\,d_1+\ldots+d_n}\delta}{\partial {x_1}^{d_1} \ldots \partial {x_n}^{d_n}}.$$

The distributions in $\mathcal{D}_0$ can be defined for very wide classes of test functions. In fact, one can take as test functions all formal power series in the coordinates $x_1, \ldots, x_n$, since they can always be evaluated at the origin, together with all their derivatives. An element of $\mathcal{D}_0$ is completely determined by its value on the formal power series. One can speak of the distributions supported at 0 in any finite-dimensional vector space V: we denote the space of such distributions by $\mathcal{D}_0(V)$.

The space $\mathcal{D}_0(V)$ depends on V functorially: a linear map of vector spaces $V \to W$ induces a map

$$\mathcal{D}_0(V) \to \mathcal{D}_0(W).$$

Moreover,

$$\mathcal{D}_0(V \oplus W) = \mathcal{D}_0(V) \otimes \mathcal{D}_0(W).$$

These two properties imply that $\mathcal{D}_0(V)$ is a commutative and cocommutative Hopf algebra. The product $\cdot$ in $\mathcal{D}_0(V)$ is induced by the map

$$V \oplus V \to V$$

$$(u, v) \mapsto u + v,$$

and the coproduct Δ by the diagonal map

$$V \to V \oplus V$$

$$u \mapsto (u, u).$$

Explicitly, the product in $\mathcal{D}_0$ is given by

$$\xi_1^{d_1} \dots \xi_n^{d_n} \cdot \xi_1^{d'_1} \dots \xi_n^{d'_n} = \xi_1^{d_1+d'_1} \dots \xi_n^{d_n+d'_n}.$$

In particular, $\mathcal{D}_0$ is generated by the ξ_i. Dirac's delta is the unit, so we will denote it simply by 1. The co-product is an algebra homomorphism, so it is defined by its value on the generators:

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i.$$

For an arbitrary V, the Hopf algebra $\mathcal{D}_0(V)$ is the *symmetric algebra* on V.

All the above formulae are readily obtained by observing how formal power series behave under linear maps.

2.2 Formal products and the convolution of distributions

The spaces $\mathcal{D}_0(V)$ behave functorially not only under linear maps of vector spaces but also under so-called *formal maps* (see, for instance, [3]). The formal maps that are of importance for us here are the *unital formal products* on $\mathbb{R}^n$. A unital formal product F on $\mathbb{R}^n$ is an n-tuple of formal power series in $2n$ variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$ of the form

$$F_i(x_1, \dots, x_n, y_1, \dots, y_n) = x_i + y_i + \sum_{j,k=1}^{n} x_j y_k \cdot f_{ijk}(x_1, \dots, x_n, y_1, \dots, y_n),$$

where $i = 1, \dots, n$ and $f_{ijk}(x_1, \dots, x_n, y_1, \dots, y_n)$ are some formal power series.

A unital formal product F on $\mathbb{R}^n$ induces a so-called *convolution* product on $\mathcal{D}_0$. If $h(x_1, \dots, x_n)$ is a formal power series, define a formal power series $F^*(h)$ in $x_1, \dots, x_n$ and $y_1, \dots, y_n$ by

$$\begin{aligned} &F^*(h)(x_1, \dots, x_n, y_1, \dots, y_n) \\ &\qquad = h(F_1(x_1, \dots, x_n, y_1, \dots, y_n), \dots, F_n(x_1, \dots, x_n, y_1, \dots, y_n)). \end{aligned}$$

Then for $\mu, \nu \in \mathcal{D}_0$, the convolution $\mu \star \nu \in \mathcal{D}_0$ is defined as

$$(\mu \star \nu)(h(x_1, \dots, x_n)) = (\mu \otimes \nu)(F^*(h)(x_1, \dots, x_n, y_1, \dots, y_n)).$$

Here, $\mu \otimes \nu$ is thought of as a distribution on the euclidean space $\mathbb{R}^n \oplus \mathbb{R}^n$ with the coordinates $x_1, \dots, x_n$, $y_1, \dots, y_n$.

The most basic unital formal product is given by $F_i(x_1, \dots, y_n) = x_i + y_i$. In this case, the convolution product coincides with the product $\mathcal{D}_0$. In general, $\mathcal{D}_0$ with the convolution product and the coproduct Δ is a not necessarily associative bialgebra (in fact, a Hopf algebra, with an appropriate definition of a non-associative Hopf algebra), see [3, 4].

Assume now that F is a formal group and the F_i are the power series expansions of the product in a Lie group G in some fixed coordinates, in a neighbourhood of the neutral element of G. Serre proves that, in this case, $\mathcal{D}_0$ with the convolution product is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. It has been observed in [3] that this same construction can be used to define universal enveloping algebras for arbitrary Sabinin algebras.

3 De Rham currents and the Chevalley-Eilenberg resolution

3.1 Currents

In the same way as Schwartz distributions are defined as continuous linear functions on spaces of test functions, *de Rham currents* are continuous linear functions on differential forms [5]. Those currents that are only non-zero on differential p-forms are called *p-dimensional*. This terminology comes from the fact that a compact oriented p-dimensional submanifold $M \subset \mathbb{R}^n$ defines a linear function on p-forms by

$$\phi \mapsto \int_M \phi.$$

Distributions are 0-dimensional currents.

We say that a current T is supported on a set $A \in \mathbb{R}^n$ if $T(\phi) = 0$ for each form ϕ that vanishes on A. We will be interested in the currents supported at the origin of $\mathbb{R}^n$. As for the test forms, we will consider "formal p-forms" of the form

$$\sum_{i_1<\ldots<i_p} f_{i_1,\ldots,i_p}(x_1,\ldots,x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p},$$

where the $f_{i_1,\ldots,i_p}$ are formal power series in $x_1,\ldots,x_n$.

Let $d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p} \in \Lambda^p(\mathbb{R}^n)$ be the p-multivector dual to $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. Define $\mathcal{D}_p$ to be the vector space of the p-dimensional currents of the form

$$\sum_{i_1<\ldots<i_p} \mu_{i_1,\ldots,i_p}\, d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p},$$

where $\mu_{i_1,\ldots,i_p} \in \mathcal{D}_0$. In other words,

$$\mathcal{D}_p = \mathcal{D}_0 \otimes \Lambda^p(\mathbb{R}^n).$$

The value of the current $\mu\, d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p}$ on the p-form

$$f(x_1,\ldots,x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

is equal to $\mu(f)$ if $i_k = j_k$ for $k = 1,\ldots,p$ and 0 otherwise.

The space

$$\mathcal{D}_* = \mathcal{D}_*(\mathbb{R}^n) = \bigoplus_p \mathcal{D}_p$$

of all currents supported at the origin is, actually, a chain complex whose boundary map ∂ is dual to the differential d on the forms. An explicit computation shows that

$$\partial\big(\mu\, d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p}\big) = \sum_{k=1}^{p} (-1)^{k+1} \mu\, \xi_{i_k}\, d\xi_{i_1} \wedge \ldots \widehat{d\xi_{i_k}} \ldots \wedge d\xi_{i_p},$$

where, as usual, $\widehat{d\xi_{i_k}}$ indicates an omitted term.

The complex $(\mathcal{D}_*, \partial)$ is acyclic since the same is true for the complex of the formal differential forms on $\mathbb{R}^n$.

3.2 $\mathcal{D}_*$ as a differential graded Hopf algebra

The space of currents enjoys the same functorial properties as the space of distributions. A choice of coordinates in a vector space V allows one to speak about the space $\mathcal{D}_*(V)$. A linear map $V \to W$ induces a map $\mathcal{D}_*(V) \to \mathcal{D}_*(W)$ and there is an isomorphism of differential graded vector spaces

$$\mathcal{D}_*(V \oplus W) = \mathcal{D}_*(V) \otimes \mathcal{D}_*(W);$$

as a consequence, $\mathcal{D}_*(V)$ is a commutative and cocommutative differential graded Hopf algebra.

The coproduct Δ in $\mathcal{D}_*$ is induced by the diagonal map and is determined by

$$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i \qquad \text{and} \qquad \Delta(d\xi_i) = d\xi_i \otimes 1 + 1 \otimes d\xi_i.$$

The (graded) commutative product is induced by that of $\mathcal{D}_*$ and that of $\Lambda^*(\mathbb{R}^n)$.

A unital formal product F on $\mathbb{R}^n$ induces a pullback map from the space of formal p-forms on $\mathbb{R}^n \oplus \mathbb{R}^n$ to the p-forms on $\mathbb{R}^n$. Dually, we obtain a convolution product

$$*_F : \mathcal{D}_* \otimes \mathcal{D}_* \to \mathcal{D}_*.$$

For $\phi \in \mathcal{D}_p$ and $\psi \in \mathcal{D}_q$, the current $\phi *_F \psi \in D_{p+q}$ is defined by its values on r-forms with $r = p + q$:

$$\begin{aligned}(\phi *_F \psi)&(f(x_1, \ldots, x_n)\, dx_{i_1} \wedge \ldots \wedge dx_{i_r}) \\ &= (\phi \otimes \psi)(f(F_1, \ldots, F_n)\, dF_{i_1} \wedge \ldots \wedge dF_{i_r}),\end{aligned}$$

where, on the right-hand side, $F_i = F_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and the tensor product $\phi \otimes \psi$ acts by ϕ on the x-coordinates and by ψ on the y-coordinates.

Theorem 3.1. *The convolution product $*_F$ together with the coproduct Δ endows $\mathcal{D}_*$ with the structure of an acyclic differential graded bialgebra. When F is a formal group with the Lie algebra $\mathfrak{g}$, $*_F$ is associative and $\mathcal{D}_*$ coincides with the Chevalley-Eilenberg resolution $\mathcal{CE}(\mathfrak{g})$.*

The first part of the theorem is a straightforward check using the definitions. It is also clear that the associativity of F implies that of $*_F$. As for the comparison with the Chevalley-Eilenberg resolution, it follows from a relatively little-known description of $\mathcal{CE}(\mathfrak{g})$ which is, apparently, due to Cartier[2].

Consider the two-term differential graded Lie algebra (the *cone on* $\mathfrak{g}$)

$$\ldots \to 0 \to \mathfrak{g} \xrightarrow{\mathrm{id}} \mathfrak{g}$$

with differential of degree -1 and copies of $\mathfrak{g}$ in degrees 0 and 1. It is acyclic and, therefore, its universal enveloping algebra is also acyclic (see [7, Appendix B, Proposition 2.1]). The Chevalley-Eilenberg resolution $\mathcal{CE}(\mathfrak{g})$ is precisely this universal enveloping algebra.

On the other hand, $\mathcal{D}_*$ with the product $*_F$ is a cocommutative bialgebra in the category of differential graded vector spaces, generated by its primitive elements ξ_i and $d\xi_i$. These primitive elements form a differential graded Lie algebra which is precisely the cone on $\mathfrak{g}$. By the Milnor-Moore theorem, $\mathcal{D}_*$ with the product $*_F$ is then isomorphic to the universal enveloping algebra of the cone on $\mathfrak{g}$, that is, to $\mathcal{CE}(\mathfrak{g})$.

Another way to think of this proof is to observe that the algebra of currents $\mathcal{D}_*$ with the convolution corresponding to a product in a Lie group G may be thought of as the algebra of distributions on the tangent bundle to G considered as a graded Lie group. On the other hand, the cone on $\mathfrak{g}$ is the Lie algebra tangent to this graded Lie group. In this way, the construction of the Chevalley-Eilenberg resolution via currents is just the graded version of Serre's result on $\mathcal{D}_0$.

4 Distribution algebras for non-associative products

One may ask what is gained by allowing non-associative products in the construction of the algebra of the de Rham currents. It turns out that non-associative products exist naturally on homogeneous spaces;

[2] I am indebted to Olivier Mathieu for this information.

we will see here how the complex of invariant forms on a homogeneous space arises from the "non-associative Chevalley-Eilenberg resolution".

4.1 Non-associative products and invariant forms

Let M be a manifold with an analytic unital non-associative product. The left multiplication

$$L_a : M \to M$$

$$y \mapsto ay$$

is bijective for a in some neighbourhood U of the unit. A differential form ω on M is *left-invariant* if $L_a^*(\omega) = \omega$ for $a \in U$.

The complex of left-invariant forms on M can be recovered from the algebra $\mathcal{D}_*$ of the de Rham currents with the convolution product. Let $\overline{\mathcal{D}}_0$ be the kernel of the counit of $\mathcal{D}_0$; it consists of distributions "with no constant term". Consider the cochain complex

$$\mathrm{Hom}_{\overline{\mathcal{D}}_0}(\mathcal{D}_*, \mathbb{R})$$

of linear functions on $\mathcal{D}_*$ that vanish on the elements of the form $\mu * \phi$ where $\phi \in \mathcal{D}_*$ and $\mu \in \overline{\mathcal{D}}_0$.

Proposition 4.1. $\mathrm{Hom}_{\overline{\mathcal{D}}_0}(\mathcal{D}_*, \mathbb{R})$ *is the complex of the left-invariant forms with respect to the product on M.*

Sketch of the proof. Consider a linear function on currents given by a p-form ω on the manifold M.

The pullback of ω from M to $M \times M$ via the multiplication map can be written as

$$L_x^*(\omega(y)) + \sum \omega_{i_k}(x) \otimes \omega_{j_k}(y),$$

where each $\omega_{i_k}(x)$ is a q-form with $q > 0$. In particular, if ω is left-invariant

$$(\mu * \phi)(\omega) = (\mu \otimes \phi)(L_x^*(\omega(y))) = (\mu \otimes \phi)(\omega(y)) = 0.$$

On the other hand, if $(\mu \otimes \phi)(L_x^*(\omega(y))) = 0$ for all $\phi \in \mathcal{D}_*$ and $\mu \in \overline{\mathcal{D}}_0$, this means that $L_x^*(\omega(y)) = \omega(y)$. □

The construction of the algebra of currents supported at the unit is, by definition, local. Therefore, the above proposition remains true if the product on M is defined only on some neighbourhood of the unit in M: in this situation left-invariant differential forms should be replaced by *germs* of left-invariant forms.

4.2 Non-associative products on homogeneous spaces

Let $H \subset G$ be a closed subgroup of a Lie group G and let $\pi : G \to G/H$ be the projection map. Let $s : G/H \to G$ be a section of the projection map, that is, a map with the property that $\pi \circ s = \mathrm{Id}$. Then, the homogeneous space G/H carries a non-associative product

$$x * y = \pi(s(x)s(y)) \tag{1}$$

whose unit is the image of the unit $e \in G$.

The section $s : G/H \to G$ may fail to exist for topological reasons. Nevertheless, one can always find a section of π over some neighbourhood $U \subset G/H$ of $\pi(e)$. In this case, (1) defines a *local* product

$$U \times U \to G/H.$$

The construction of the algebra of currents supported at the unit is, by definition, local, and we see that any homogeneous space gives rise to a non-associative differential graded algebra of currents $\mathcal{D}_*(G/H)$. Note that the product in this algebra depends on the section s.

Proposition 4.2.

$$\mathrm{Hom}_{\overline{\mathcal{D}}_0(G/H)}(\mathcal{D}_*(G/H), \mathbb{R}) = \Lambda^*(\mathfrak{g}/\mathfrak{h})^{\mathfrak{g}},$$

where $\mathfrak{g}$ *and* $\mathfrak{h}$ *are the Lie algebras of* G *and* H *respectively.*

Sketch of the proof. We have to verify that the left-invariant forms on G/H are the ones that are left-invariant with respect to the product (1). For simplicity, assume that the left products by the elements in the image of s generate G.

Consider a form, defined on the image of the section s, left-invariant with respect to (1). Such forms are in one-to-one correspondence with the forms ω on G, right-invariant with respect to H and such that

$$\omega = L_a^*\omega$$

for a in the image of s. Since for any $g \in G$ we have

$$L_g^* = L_{a_1}^* \ldots L_{a_k}*$$

for some k, we see that ω is left G-invariant. $\square$

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References

[1] N. Bourbaki, Éléments de mathématique. Fasc. XXXVII. Groupes et algbres de Lie. Chapitre II: Algbres de Lie libres. Chapitre III: Groupes de Lie. Hermann, Paris, 1972.

[2] O. Loos, Symmetric Spaces: Vol.: 1 : General Theory, W.A. Benjamin, 1969.

[3] J. Mostovoy, J.M. Pérez Izquierdo, Formal multiplications, bialgebras of distributions and non-associative Lie theory. Transform. Groups **15** (2010) 625–653.

[4] J. Mostovoy, J.M. Pérez Izquierdo, I. Shestakov, Hopf algebras in non-associative Lie theory, Bull. Math. Sci. **4** (2014) 129–174.

[5] G. de Rham, Variétés differentiables. Formes, courants, formes harmoniques. Hermann, Paris, 1956.

[6] J.-P. Serre, Lie algebras and Lie groups, W. A. Benjamin, Inc., New York-Amsterdam, 1965.

[7] D. Quillen, Rational homotopy theory, Ann. of Math. **90** (1969) 205–295.