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Round twin groups*

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Abstract

We study the fundamental group of the configuration space of n ordered points on the circle no three of which are equal. We compute it for $n < 6$ and describe its mod 2 homology for $n = 6$. We also show how, for arbitrary n, this group can be assembled from planar braid groups and relate it to the pure cactus group.

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1 Introduction

The pure braid group on n strands, defined as the fundamental group of the space of configurations of n distinct points in $\mathbb{C}$, has many generalizations. The most straightforward way to produce other braid groups is to replace the complex plane with another surface (a manifold of real dimension 2). Configuration spaces on higher-dimensional simply-connected manifolds are simply-connected; however, they also produce braid-like groups if, instead of the fundamental groups one considers other homotopy functors as in [4]. Configuration spaces in 1-dimensional manifolds seem to be less appealing: indeed, the configuration space of n distinct ordered points in $\mathbb{R}$ consists of $n!$ contractible pieces. Nevertheless, this space has several naturally defined (partial) compactifications whose fundamental groups turn out to be of interest.

Two of these "real versions of the braid groups" have been studied in some detail. The planar braid group, also known as the twin group, is the fundamental group of the space of n-tuples of particles in $\mathbb{R}$ no three of which are allowed to coincide. Its elements can be represented

*Invited paper.

by n-tuples of descending strands in a horizontal strip without triple intersections. Planar braid groups appeared in various contexts; in particular, in the theory of so-called doodles [15], or in physics in the study of three-body interactions [10].

Another closely related group is the cactus group [11]. Its "pure" version is the fundamental group of the moduli space of stable real rational curves with n marked points. Although elements of the cactus group are not usually though of as braids, such a representation exists and may be useful in some situations, see [17, 20].

In this note we observe that there is yet another group that deserves to be considered as the "real version" of the pure braid group and fits between pure planar braids and pure cactus groups. It is the fundamental group of the configuration space of n ordered points on the circle $S^1 = \mathbb{R} \cup \{\infty\}$ no three of which are allowed to coincide, and such that the nth point lies at infinity. We call it the *round twin group on n strands.* Here, we will identify the round twin groups on up to 5 strands and describe the round twin group on 6 strands as the fundamental group of a link complement in a certain 3-manifold. We also show how round twin groups are assembled from pure planar braid groups and exhibit a long exact sequence involving their cohomology.

1.1 Planar braids on a line

The *twin group* $\overline{B}_n$, or the group of *planar braids on n strands*, has a presentation with the generators by $\sigma_1, \ldots, \sigma_{n-1}$ and the relations

$$\begin{aligned} \sigma_i^2 &= 1 && \text{for all } 1 \leq i < n; \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for all } 1 \leq i, j < n \text{ with } |i-j| > 1. \end{aligned}$$

There is a homomorphism of $\overline{B}_n$ onto the symmetric group S_n which sends the generator σ_i to the transposition $(i\ i+1)$. The kernel of this homomorphism is called the *pure twin group* or the *planar pure braid group* $\overline{P}_n$. It is the fundamental group of the configuration space $M_n = M_n(\mathbb{R})$ of n ordered particles in $\mathbb{R}$ no three of which are allowed to coincide.

The pure twin groups on 3, 4 or 5 strands are free on 1, 7 and 31 generators respectively, while the pure twin group on 6 strands is a free product of 71 copies of the infinite cyclic group and 20 copies of the free abelian group on 2 generators, see [18]. In general, the group $\overline{P}_n$ has a minimal presentation whose relations are commutators. The cohomology ring of $\overline{P}_n$ is known (see [1, 6]).

1.2 On the terminology

The groups of planar braids were discovered independently several times and were given various names: Grothendieck cartographical groups [21], twin groups [14, 15], groups of flat braids [16], traid groups [10]. The most descriptive of those, namely, "flat braids", has become inoperative after being used for a different object in the theory of virtual knots. The term "planar braids" used in [18, 19] is an attempt to produce the closest replacement to "flat braids". Unfortunately, it does not generalize too well to the situation considered in the present note, namely, that of planar braids drawn in an annulus. One might want to call them "annular braids"; as observed in [7], they form annular diagram groups in the terminology of [8]. However, the term "annular braids" has already been used for something entirely different, see [13]. For this reason, we will mostly use Khovanov's terminology of "twin groups" and refer to the elements of the corresponding groups as "twins".

1.3 Twins on a circle

One may consider configuration spaces of points on a circle rather than a line. These lead to *annular* twin groups.

The annular pure twin group $\overline{P}_n(S^1)$ is the fundamental group of the configuration space $M_n(S^1)$ of n ordered particles in S^1 no three of which may coincide. The space $M_n(S^1)$ is an open subset in the n-dimensional torus $(S^1)^n$; its complement consists of the points $(z_1, \ldots, z_n)$ which satisfy

$$z_i = z_j = z_k$$

for some triple of distinct indices i, j, k.

The *full* annular twin group $\overline{B}_n(S^1)$ has a presentation with the generators by $\alpha_1, \ldots, \alpha_n$ and η and the relations

$$\begin{array}{rcll} \alpha_i^2 &=& 1 & \text{for all } 1 \leq i \leq n; \\ \alpha_i\alpha_j &=& \alpha_j\alpha_i & \text{for all } 1 \leq i, j \leq n \text{ with } i \not\equiv j \pm 1 \mod n; \\ \alpha_i\eta &=& \eta\alpha_j; & \text{where } j = i + 1 \mod n. \end{array}$$

Annular twins can be drawn on a vertical cylinder (which, topologically, is an annulus) in the same way as the planar braids are drawn in a plane, namely, as collections of descending strands, see Figure 1. It is immediately clear that, with the help of the last relation, one can write a presentation for $\overline{B}_n(S^1)$ which only has two generators.

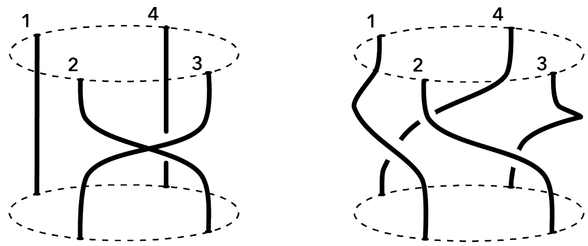

Figure 1: Generators of $\overline{B}_4(S^1)$: α_2 on the left and η on the right.

There is a homomorphism $\overline{B}_n(S^1) \to S_n$ sending α_i to $(i\ i+1)$ for $i < n$ and α_n to $(1\ n)$, and η to $(1\ 2\ 3\ \dots\ n)$. The kernel of this homomorphism is precisely the annular pure twin group $\overline{P}_n(S^1)$. Note that while the symmetric group S_n acts on $M_n(S^1)$ permuting the labels of the particles, the full annular twin group $\overline{B}_n(S^1)$ is *not* the fundamental group of $M_n(S^1)/S_n$, since this action is not free. (In fact, $M_n(S^1)/S_n$ is easily seen to be simply connected).

Instead of the annular twin groups, it may be convenient to consider their subgroups that consist only of those twins whose nth strand is vertical. Then, if we think of S^1 as $\mathbb{R} \cup \{\infty\}$, this strand can be placed at the infinity and the twin can be drawn in a plane as in Figure 2. We

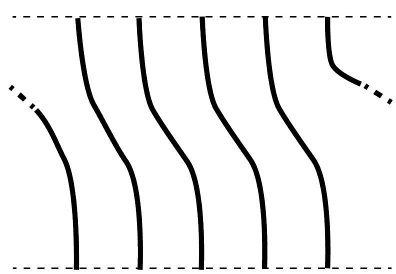

Figure 2: The generator ζ of $\Upsilon_5 \subset \overline{B}_6(S^1)$.

call such annular twins *round twins*. The subgroups of all round twins in $\overline{B}_n(S^1)$ and $\overline{P}_n(S^1)$ will be denoted by Υ_{n-1} and Π_n respectively. (The mismatch in the indices has its origin in the standard notation for cactus groups, see Section 3).

A presentation for Υ_n is easy to obtain from the presentation for $\overline{B}_{n+1}(S^1)$. For, instance, Υ_n can be given by the generators $\sigma_1, \dots, \sigma_{n-1}$

and ζ subject to the relations

$$(1)\qquad \begin{array}{rcll} \sigma_i^2 & = & 1 & \text{for all } 1 \leq i < n; \\ \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for all } 1 \leq i, j < n \text{ with } |i-j| < 1; \\ \sigma_i\zeta & = & \zeta\sigma_{i+1}; & \text{for all } 1 \leq i < n-1. \end{array}$$

In terms of the generators of $\overline{B}_{n+1}(S^1)$, we have $\sigma_i = \alpha_i$ for $1 \leq i < n$ and $\zeta = \alpha_n\eta$. As in the case of $\overline{B}_{n+1}(S^1)$, one can also write a presentation with two generators only. We do not have a good presentation for the pure round twin group.

As mentioned in the introduction, it may be tempting to think of twins and annular twins as "real versions" of usual braids, which are paths of configurations of points in a complex plane. From this point of view, the group Υ_n of round twins may be a good analogue of the braid group B_n. Indeed, we can think of braids on n strands in $\mathbb{C}$ as braids on $n+1$ strands in $\mathbb{C} \cup \{\infty\}$, whose $n+1$st strand is vertical at infinity.

The relationship between the pure annular twin groups and pure round twin groups is straightforward. For each n, the configuration space $M_n(S^1)$ splits as a Cartesian product $S^1 \times Q_n$, where $Q_n \subset M_n(S^1)$ is the subspace consisting of the points with $z_n = \infty$. As a consequence, we have

$$\overline{P}_n(S^1) = \mathbb{Z} \times \Pi_n$$

for all n since $\Pi_n = \pi_1 Q_n$.

For low values of n the pure round twin groups can be described explicitly as follows:

Theorem 1.1. *The group* Π_1 *is trivial. We have*

$$\begin{aligned} \Pi_2 &= \mathbb{Z}, \\ \Pi_3 &= F_2, \\ \Pi_4 &= F_4, \\ \Pi_5 &= \pi_1 X_4, \end{aligned}$$

where X_4 *is the Riemann surface of genus 4 and* F_k *stands for the free group on* k *generators.*

When $n \leq 4$ this statement is easy to verify directly from the definition. For $n = 5$, it can be deduced from the fact that Q_5 is a disk bundle over the orienting double cover of the moduli space $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ of stable real rational curves with 5 marked points; $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is a connected sum of 5 real projective planes.

The connection with the moduli spaces comes from the fact that the space Q_n is homotopy equivalent to the quotient $M_n(S^1)/SL(2,\mathbb{R})$, with the natural action of $SL(2,\mathbb{R})$ on $S^1 = \mathbb{R}P^1$. This connection also gives us the following result:

Theorem 1.2. *The group* Π_6 *is the fundamental group of the complement of a 10-component link in the 3-manifold which is the boundary of a 4-disk with five 1-handles.*

This fact is a direct consequence of the known description of $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$ as a blowup of a certain configuration of 5 points and 10 lines in $\mathbb{RP}^3$. In principle, one might use it to compute the cohomology of Π_6; we will find the mod 2 Betti numbers of Π_6 in Section 5 by other methods.

In general, we will show how to assemble Π_n from the planar pure braid groups $\overline{P}_k$ with the tools of Bass-Serre theory.

Theorem 1.3. Π_n *is the fundamental group of a graph of groups with* n *vertices: one vertex labelled with* $\overline{P}_{n-1}$ *and* $n-1$ *vertices labelled with* $\overline{P}_{n-2}$.

It follows from this statement that Q_n is an Eilenberg-MacLane space and, therefore, its cohomology coincides with that of Π_n. The description of Q_n in terms of the groups $\overline{P}_k$, in principle, gives a way to compute the cohomology of Q_n, since the cohomology of $\overline{P}_k$ is known completely. Nevertheless, this is not a straightforward task and we illustrate it on the example of Q_6.

The note has the following structure. In the next section we prove Theorems 1.1 and 1.2. We will not give any introduction to the moduli spaces of stable rational curves referring the reader instead to [5, 11, 12]. In Section 3 we compare the round twin groups to the cactus groups, that is, the fundamental groups of the moduli spaces of curves. Namely, we show that the round twin group is a subgroup of the corresponding full cactus group. In Section 4 we show that Π_n is the fundamental group of a certain graph of planar braid groups. Finally, in Section 5 we obtain a long exact sequence for the cohomology of Π_n and compute the groups $H^*(\Pi_6, \mathbb{Z}_2)$.

2 Round twins with few strands

2.1 Π_1, Π_2 and Π_3

The group Π_1 is trivial and $\Pi_2 = \mathbb{Z}$ since Q_1 is a point and $Q_2 = S^1$. The space Q_3 is the punctured torus $(z_1, z_2) \neq (\infty, \infty)$ and, therefore, $\Pi_3 = F_2$.

2.2 The group Π_4

As for Q_4, it is the complement in the torus $(S^1)^3$ to the union of the sets (∞, ∞, t), (∞, t, ∞), (t, ∞, ∞) and (t, t, t) with $t \in S^1$. It is shown in Figure 3 as a fundamental region in its universal cover; one has to identify the opposite faces of the cube and remove the black lines. Q_4

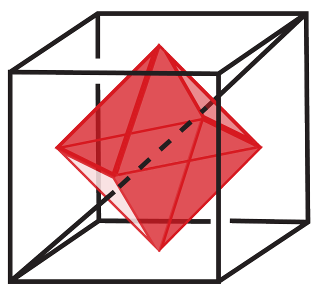

Figure 3: The space Q_4.

can be retracted onto the 2-dimensional subcomplex of $(S^1)^3$ which is an octahedron with two opposite faces removed and opposite vertices identified; its fundamental group is F_4.

2.3 Π_5 and the moduli space of real stable rational curves with 5 marked points

The group $PSL(2, \mathbb{R})$ acts on S^1 and, hence, on the space $M_n(S^1)$, by real Möbius transformations. The quotient space $M_n(S^1)/PSL(2, \mathbb{R})$ is, in fact, a subspace of the moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of stable real rational curves with n marked points. This space is very well-studied; we refer to [5, 12] for a detailed description of its geometry and combinatorics. A point in $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ is a tree of projective lines with n marked points, whose components have no automorphisms, considered up to a Möbius transformation on each component. A point in this moduli space lies in $M_n(S^1)/PSL(2,\mathbb{R})$ if it corresponds to a curve whose graph of components is a star, that is, has all but one vertices univalent, and whose every component represented by a univalent vertex has exactly 2 marked points on it, see Figure 4.

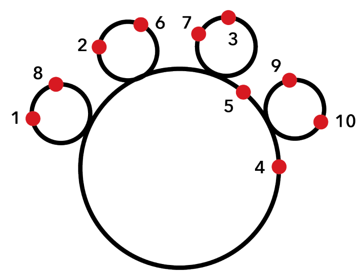

Figure 4: A point in $\overline{\mathcal{M}}_{0,10}(\mathbb{R})$ coming from Q_{10}.

In general, $M_n(S^1)/PSL(2,\mathbb{R})$ is the complement to a codimension 1 subset of $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$. However, when $n = 5$, each point of $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ comes from $M_5(S^1)$. In fact, the map

$$M_5(S^1) \to \overline{\mathcal{M}}_{0,5}(\mathbb{R})$$

factors as

$$M_5(S^1) \to \mathcal{M}_5(S^1)/SL(2,\mathbb{R}) \to M_5(S^1)/PSL(2,\mathbb{R}) = \overline{\mathcal{M}}_{0,5}(\mathbb{R}),$$

where the first map, up to homotopy, is a trivial circle bundle and the second map is the orienting double cover. Since $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is a connected sum of five real projective planes, it follows that $M_5(S^1)/SL(2,\mathbb{R})$ is a Riemann surface of genus 4. On the other hand, $M_5(S^1)/SL(2,\mathbb{R})$ is homeomorphic to Q_5.

Remark 2.1. The fact that $\Pi_5 = \pi_1 X_4$ can also be established by means of Corollary 4.2 in Section 4.

2.4 Π_6 and $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$

The complement to $M_6(S^1)/PSL(2,\mathbb{R})$ in $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$ is the closure of the subset of all curves with two components, with three marked points on each component. The combinatorial types of all the curves in this closure are shown in Figure 5. In fact, $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$ is the blowup of $\mathbb{R}P^3$ along the configuration of points and lines shown in Figure 6; first, one

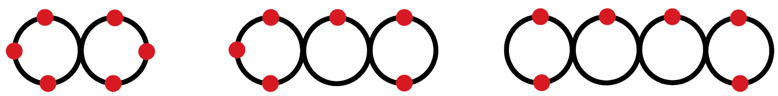

Figure 5: Points in $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$ not coming from $M_6(S^1)$.

blows up $\mathbb{R}P^3$ at the 5 points and then at the 10 lines connecting them. The complement of $M_6(S^1)/PSL(2,\mathbb{R})$ is then precisely the exceptional divisor of the blowup along the 10 lines, since this exceptional divisor consists of the curves on Figure 5; see [5].

This means that $M_6(S^1)/PSL(2,\mathbb{R})$ is a complement to a 10-component link in the blowup of $\mathbb{R}P^3$ at 5 points. The orienting double cover of the blowup of $\mathbb{R}P^3$ at 5 points consists of two 3-spheres with 6 punctures connected by six 1-tubes; this is readily seen to be the boundary of a 4-disk with five 1-handles. The 10 components of the exceptional divisor lift to 10 circles in this manifold. The space $M_6(S^1)/PSL(2,\mathbb{R})$ can be identified with the complement to these circles and Q_6 is homeomorphic to it. This establishes Theorem 1.2.

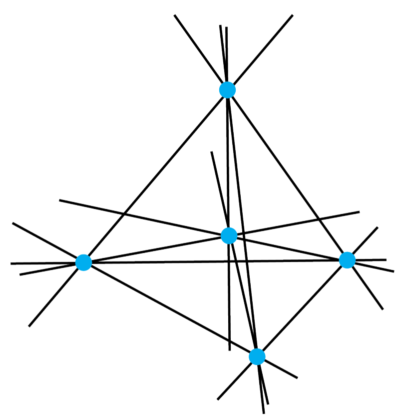

Figure 6: The configurations of points and lines in $\mathbb{R}P^3$ whose blowup is $\overline{\mathcal{M}}_{0,6}(\mathbb{R})$.

3 Relationship with the cactus groups

The group

$$\Gamma_n = \pi_1 \overline{\mathcal{M}}_{0,n}(\mathbb{R})$$

is known as the nth *pure cactus group*. The map

$$Q_n \simeq M_n(S^1)/SL(2,\mathbb{R}) \to \overline{\mathcal{M}}_{0,n}(\mathbb{R})$$

gives rise to a homomorphism

$$\Pi_n \to \Gamma_n$$

for each n. While we do not have neat presentations for either of the two series of groups, this homomorphism can be described very explicitly in terms of the corresponding full groups.

The *full* cactus group J_n (see [11]) has a presentation with the generators $s_{p,q}$, where $1 \leq p < q \leq n$, and the following relations:

$$\begin{array}{rcll} s_{p,q}^2 & = & 1, & \\ s_{p,q}s_{m,r} & = & s_{m,r}s_{p,q} & \text{if } [p,q]\cap[m,r]=\emptyset, \\ s_{p,q}s_{m,r} & = & s_{p+q-r,p+q-m}s_{p,q} & \text{if } [m,r]\subset[p,q]. \end{array} \tag{2}$$

There is a homomorphism $J_n \to S_n$ to the symmetric group: it sends $s_{p,q}$ into the permutation $\tau_{p,q}$ of $\{1,\dots,n\}$ which reverses the order of $p, p+1, \dots, q$ and leaves the rest of the elements unchanged. The pure cactus group Γ_{n+1} is the kernel of this homomorphism.

Consider the homomorphism

$$\kappa : \Upsilon_n \to J_n$$

defined by

$$\begin{array}{rcl} \kappa(\sigma_i) & = & s_{i,i+1}, \\ \kappa(\zeta) & = & s_{1,n}s_{2,n}. \end{array}$$

with σ_i and ζ as in (1). This homomorphism is clearly well-defined and sends $\Pi_{n+1} \subset \Upsilon_n$ to Γ_{n+1}. We will denote by the same letter the map sending words in the generators of Υ_n to words in the generators of J_n.

In order to see that the restriction of κ to Π_n is actually induced by the map

$$\Pi_n \to \overline{\mathcal{M}}_{0,n}(\mathbb{R}),$$

one has to recall the geometric meaning of the generators of J_n: the generator $s_{p,q}$ corresponds to a path in $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ in which the marked points $p,\dots,q$ collide and bubble off onto a new component and then return to the original component in the reversed order (see [11]). This shows that $\sigma_{i,i+1}$ map into $s_{i,i+1}$ and ζ must go to $s_{1,n}s_{2,n}$.

It is not hard to see that κ is not always injective; for instance, $\Pi_4 = F_4$ while Γ_4 is infinite cyclic.

Proposition 3.1. *The homomorphism $\kappa : \Upsilon_n \to J_n$ is injective for $n \geq 4$.*

Proof. If w is a word in the generators $s_{p,q}$ that defines the trivial element of J_n, there exists a sequence $w_1, \dots, w_m$ such that $w_1 = w$, w_n is trivial, w_{i+1} is obtained from w_i by applying one of the relations (2) once and the length of w_{i+1} is not greater than the length of w_i.

This follows from the proof of Proposition 2 in [17] where it is shown that two words in the $s_{p,q}$ which represent the same element of J_n and are *locally reduced* (their lengths cannot be decreased by applying the relations (2)) must have the same length. Indeed, if w can only be taken into the trivial word by a sequence of moves that increases the length at some point, there exists a locally reduced word which represents the trivial element in J_n, which is impossible.

The image of a word in the generators σ_i and ζ under κ is a word w in the $s_{i,i+1}$, $s_{1,n}$ and $s_{2,n}$. If it represents the trivial element of J_n, it can be transformed into the trivial word by means of the relations that involve only the generators $s_{i,i+1}$, $s_{1,n}$, $s_{2,n}$ and $s_{1,n-1}$ only, since the appearance of any other generator of J_n in the sequence of the words connecting w and 1 would imply that at some point the length of the word increases.

Assume that $n \geq 4$; under this condition neither of $s_{1,n}$, $s_{1,n-1}$ or $s_{2,n}$ coincides with any of $s_{i,i+1}$. Let z be the word $s_{1,n}s_{2,n}$. For any word in u in the generators $s_{i,i+1}$, $s_{1,n}$, $s_{2,n}$ and $s_{1,n-1}$, define the word $\mu(u)$ in the $s_{1,2}$, $\dots$, $s_{n-1,n}$, z and $s_{1,n}$ inductively as follows.

For a word w in the generators $s_{p,q}$, et $\overline{w}$ be the word in which each $s_{p,q}$ is replaced by $s_{n-q+1,n-p+1}$. Now:

- if $u = 1$ we set $\mu(u) = 1$;
- if $u = s_{i,i+1}v$, we set $\mu(u) = s_{i,i+1}\mu(v)$;
- if $u = s_{1,n}v$, we set $\mu(u) = \overline{\mu(v)}s_{1,n}$;
- if $u = s_{2,n}v$, we set $\mu(u) = z^{-1}\overline{\mu(v)}s_{1,n}$;
- if $u = s_{1,n-1}v$, we set $\mu(u) = z\overline{\mu(v)}s_{1,n}$.

For any word u in $s_{i,i+1}$, $s_{1,n}$, $s_{2,n}$ and $s_{1,n-1}$, the word $\mu(u)$ is of the form $\mu'(u)s_{1,n}^k$, where $\mu(u)$ is a word in $s_{i,i+1}$, and z only.

Assume v is a word in the generators σ_i and ζ such that $\kappa(v)$ defines a trivial element of J_n. Take the sequence of words $w_1, \dots, w_n$ in the $s_{p,q}$, such that $w_1 = \kappa(v)$, w_m is trivial, w_{i+1} is obtained from w_i by applying one of the relations (2) once and the length of w_{i+1} is not greater than the length of w_i. Then, the sequence of words $\mu'(w_1), \dots, \mu'(w_n)$ in $s_{1,2}$, $\dots$, $s_{n-1,n}$ and z transforms $\kappa(v)$ into the trivial word by means of the relations (2). Replacing each $s_{i,i+1}$ with σ_i and z with ζ, we obtain a sequence of words that transforms v into the trivial word by means of the relations (1) in Υ_n. In particular, this means that $v = 1$ in Υ_n. □

Remark 3.2. The fact that the twin groups inject into the cactus groups has been observed in [2].

4 Round braids via graphs of groups

Let $n > 2$ and assume that the points of the configurations in M_{n-1} are labelled by the natural numbers from 1 to $n-1$. Denote by $M_{n-2,j}$ a copy of M_{n-2} whose configurations are labelled by natural numbers from 1 to $n-1$ with the label j omitted.

For a configuration $x \in M_{n-2,j}$ define $\rho_j(x) \in M_{n-1}$ by adding a point x_j with the label j to the right of all the points of x, Similarly, $\lambda_j(x)$ is defined by adding x_j to the left of x. These concatenation operations can be considered as maps

$$M_{n-2,j} \to M_{n-1}.$$

Indeed, in both cases one can assume that all the points of each configuration in $M_{n-2,j}$ lie in some fixed open interval and choose x_j to be a fixed point outside of this interval. Note that, in general, these maps do not preserve the basepoints.

In the union

$$M_{n-1} \sqcup \bigsqcup_{1 \le j < n} M_{n-2,j} \times [-1,1],$$

identify, for each $x \in M_{n-2,j}$, the point $(x, -1)$ with $\lambda_j(x) \in M_{n-1}$ and the point $(x, 1)$ with $\rho_j(x) \in M_{n-1}$. Denote the resulting space by Q'_n.

Theorem 4.1. *The space Q'_n is homotopy equivalent to Q_n.*

This result, which will be proved towards the end of this subsection, allows us to express Π_n as the fundamental group of a graph of groups involving $\overline{P}_{n-1}$ and $\overline{P}_{n-2}$.

For each j between 1 and $n-1$, choose a braid $g_{j+} \in \overline{B}_{n-1}$ whose permutation sends $12\dots(n-1)$ to $12\dots\hat{j}\dots(n-1)j$, and a braid g_{j-}

which sends sends $12\ldots(n-1)$ to $j12\ldots\hat{j}\ldots(n-1)$. Let $i : \overline{P}_{n-2} \to \overline{P}_{n-1}$ be the inclusion map which adds one disjoint strand on the right. Define

$$(\rho_j)_* : \overline{P}_{n-2} \to \overline{P}_{n-1}$$

as

$$x \mapsto g_{j+} x g_{j+}^{-1}$$

and, similarly, let

$$(\lambda_j)_* : \overline{P}_{n-2} \to \overline{P}_{n-1}$$

be the map

$$x \mapsto g_{j-} x g_{j-}^{-1}.$$

Now, define the graph of groups Φ_n in the following manner. The underlying directed graph of Φ_n has n vertices: one "central" vertex and $n-1$ "peripheral" vertices, with two edges from each of the peripheral vertices to the central vertex. The central vertex is labelled by $\overline{P}_{n-1}$ and the peripheral vertices by $\overline{P}_{n-2}$. Enumerate the copies of $\overline{P}_{n-2}$ from 1 to $n-1$; then, the edges emanating from the jth copy of $\overline{P}_{n-2}$ are labelled by $(\rho_j)_*$ and $(\lambda_j)_*$.

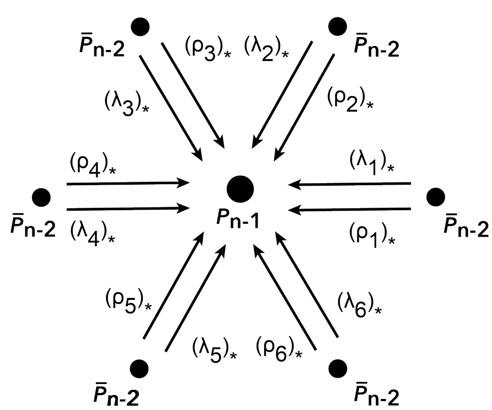

Figure 7: The graph of groups Φ_n for $n = 6$.

Corollary 4.2. *The classifying space $B\Phi_n$ for the graph of groups Φ_n is homotopy equivalent to Q_n.*

According to Theorem 1B.11 of [9], this implies that the spaces Q_n have the homotopy type of the Eilenberg-Maclane spaces $K(\Pi_n, 1)$.

Proof of Theorem 4.1. Let $Q_{n,I} \subset Q_n$ be the subspace consisting of the configurations which have either one or two points outside of a finite open interval $I \subset \mathbb{R} \subset S^1$; that is, zero or one points in $\mathbb{R}\backslash I$. We claim that the inclusion of $Q_{n,I}$ into Q_n is a homotopy equivalence. Indeed, for any pair of finite open intervals $I \subseteq I'$, the inclusion $Q_{n,I} \to Q_{n,I'}$ is a homotopy equivalence. Any compact subspace of Q_n lies in $Q_{n,I}$ for some finite interval I, and, therefore, the inclusion $Q_{n,I} \to Q_n$ induces an isomorphism of homotopy groups. Since both $Q_{n,I}$ and Q_n have homotopy type of cell complexes, they are homotopy equivalent.

The space $Q_{n,I}$ is covered by n open subspaces: U_j with $1 \leq j < n$ and V. The subspace U_j consists of the configurations whose intersection with $S^1\backslash I$ consists of the point z_j and $z_n = \infty$; the subspace V consists of those configurations whose only point at ∞ is z_n.

V is homotopy equivalent to M_{n-1} while U_j is homeomorphic to $M_{n-2} \times [-1, 1]$. The sets U_j are disjoint, while $U_j \cap V$ can be identified with two copies of M_{n-2}. The inclusion map

$$U_j \cap V \to U_j$$

is equivalent to the inclusion

$$M_{n-2} \times \{-1\} \sqcup M_{n-2} \times \{1\} \to M_{n-2} \times [-1, 1]$$

of the bases into the cylinder. The map

$$U_j \cap V \to V$$

is equivalent to the concatenation

$$x \mapsto \rho_j(x)$$

on one copy of U_j and to

$$x \mapsto \lambda_j(x)$$

on the other copy. This establishes the Theorem. □

Proof of Corollary 4.2. The construction of the space Q'_n is very similar to that of the graph of groups Φ_n. Indeed, M_k has the homotopy type of $K(\overline{P}_k)$. The mapping cylinders corresponding to the pair of edges emanating from each peripheral vertex are glued together into a cylinder of the form $M_{n-2} \times [-1, 1]$.

The braids g_{j+} and g_{j-} can be seen as paths from the basepoint of V to the basepoints of the connected components of $U_j \cap V$; denote these path by γ_{j+} or γ_{j-} respectively. Then, the maps $(\rho_j(x))_*$ and $(\lambda_j(x))_*$ are of the same form: they send the homotopy class of a path α to that of $\gamma_{j\pm}^{-1}\alpha\gamma_{j\pm}$. This shows that the fundamental group of Q'_n is precisely the fundamental group of $B\Phi_n$. □

5 The mod 2 cohomology of Π_6

In this section we compute the mod 2 cohomology groups of Π_6. Although the method that we use is, in principle, applicable to round twin groups on any number of strands, it does not produce the cup product and the combinatorial problem involved appears hard for $n > 6$. We use mod 2 coefficients for simplicity.

5.1 The cohomology of $M_n(\mathbb{R})$

The Betti numbers of M_n were found by Björner and Welker [3] and the cohomology ring was computed by Baryshnikov in the unpublished preprint [1]; a brief exposition of his work can be found in [6]. Here, for simplicity, we work with the mod 2 cohomology. Baryshnikov described the cohomology of M_n in terms of certain partially ordered sets that he called *string posets*. We will follow Baryshnikov's description although with a somewhat different terminology.

Define a *k-crossing type on n strands* as a partition of the set $\{1, \dots, n\}$ into $2k+1$ disjoint subsets $I_0, \dots, I_1, \dots, I_{2k}$ with $|I_{2i-1}| = 2$ for all $1 \leq i \leq k$. We will denote this crossing type by $(I_0, \dots, I_{2k})$ or simply by (I); the subsets I_{2i-1} will be called *crossings*. Each configuration in M_n defines a crossing type: the crossing I_{2i-1} consists of the labels of the ith (from left to right) pair of coinciding points of the configuration, and the subset I_{2i} consists of the labels of the points lying between the ith and the $i+1$st pair of coinciding points. This term "crossing type" has the following explanation: a planar pure braid is a path in M_n and each value of the parameter along this path that defines a k-crossing type with $k > 0$ corresponds to a crossing point of the strands of the braid. The number k here is the number of crossings that the braid has at this value of the parameter.

A k-crossing type (I) defines a submanifold $\zeta_{(I)}$ without boundary in M_n, of codimension k. A configuration $x \in M_n$ lies in $\zeta_{(I)}$ if and only if

(a) $x_p = x_q$ whenever $p, q \in I_{2i-1}$, and

(b) $x_p < x_q$ whenever $p \in I_i$ and $q \in I_j$ with $i < j$.

The intersection number with $\zeta_{(I)}$ determines a cohomology class in $H^k(M_n, \mathbb{Z}_2)$ which we denote simply by (I). The classes (I) are not linearly independent. In degree 1, all the relations are of the following form. For each partition of $\{1, \dots, n\}$ into three disjoint subsets I_0, I_1, I_2 with $I_1 = \{q\}$, where $1 \leq q \leq n$, we have

$$\sum_{p \in I_0} (I_0 \backslash \{p\}, \{p, q\}, I_2) = \sum_{p \in I_2} (I_0, \{p, q\}, I_2 \backslash \{p\}). \tag{3}$$

In a k-crossing type on n strands, the crossing I_{2i-1} can be *resolved* so as to obtain a $k-1$-crossing type on n strands. Namely, given a k-crossing type (I), we say that the $k-1$-crossing type (J) defined as

$$\begin{array}{lcll} J_p & = & I_p, & \text{if } p \leq 2i-3, \\ J_{2i-2} & = & I_{2i-2} \cup I_{2i-1} \cup I_{2i}, & \\ J_p & = & I_{p+2}, & \text{if } p \geq 2i-1, \end{array}$$

is obtained by resolving the ith crossing of (I).

For any two crossing types (I) and (J), the intersection of the corresponding submanifolds $\zeta_{(I)} \cap \zeta_{(J)}$ is either empty or corresponds to a crossing type which we denote by $(I) \cap (J)$. The intersection of a k-crossing type and an m-crossing type is a $k+m$-crossing type; (I) and (J) can be obtained from $(I) \cap (J)$ by resolving two complementary sets of crossings.

Theorem 5.1. *The cohomology $H^*(M_n, \mathbb{Z}_2)$ is additively generated by the crossing types on n strands; as a ring, it is generated by the 1-crossing types with the intersection product, modulo the relations* (3)*.*

Call a k-crossing type (I) *essential* if for any i between 1 and k the maximal element of the set $I_{2i-1} \cup I_{2i}$ lies in I_{2i}.

Theorem 5.2. *Essential crossing types on n strands are linearly independent and span the mod 2 cohomology of M_n.*

5.2 The long exact sequence for the cohomology of Q_n

The inclusion map $M_{n-1} \to Q_n$ can be deformed so as to obtain a cofibration

$$M_{n-1} \to Q_n \to \Sigma \left(* \sqcup \bigsqcup_{1 \leq j < n} M_{n-2,j} \right),$$

where * is a one-point space and Σ denotes the suspension. This cofibration gives rise to a long exact sequence in cohomology

$$\ldots \leftarrow \widetilde{H}^{k+1}\left(\Sigma\left(*\sqcup\bigsqcup_{1\leq j<n} M_{n-2,j}\right),\mathbb{Z}_2\right) \leftarrow H^k(M_{n-1},\mathbb{Z}_2) \leftarrow H^k(Q_n,\mathbb{Z}_2) \leftarrow \widetilde{H}^k\left(\Sigma\left(*\sqcup\bigsqcup_{1\leq j<n} M_{n-2,j}\right),\mathbb{Z}_2\right) \leftarrow \ldots,$$

which, in view of the suspension isomorphism, translates into

$$\ldots \leftarrow \bigoplus_{1\leq j<n} H^k(M_{n-2,j},\mathbb{Z}_2) \stackrel{d}{\leftarrow} H^k(M_{n-1},\mathbb{Z}_2) \leftarrow H^k(Q_n,\mathbb{Z}_2) \leftarrow \bigoplus_{1\leq j<n} H^{k-1}(M_{n-2,j},\mathbb{Z}_2) \leftarrow \ldots$$

The connecting map d sends a class c to

$$(\lambda_1^*(c)+\rho_1^*(c),\ldots,\lambda_{n-1}^*(c)+\rho_{n-1}^*(c)).$$

It follows from the description of the cohomology classes in terms of crossing types that a crossing type $(I)\in H^k(M_{n-1},\mathbb{Z}_2)$ is sent by the induced map λ_j^* to zero if $j\notin I_0$; when $j\in I_0$, it is sent to the crossing type obtained from (I) by erasing j from I_0. Similarly, ρ_j erases j from I_{2k} whenever $j\in I_{2k}$ and sends (I) to zero if $j\notin I_{2k}$.

Although we have an exact description of the cohomology of the spaces M_k, it does not seem to be a straightforward task to compute the cohomology of Q_n in general. In the following subsection we compute the mod 2 cohomology of Q_6.

5.3 Six strands

Here, we will prove the following

Theorem 5.3. *The non-trivial mod 2 cohomology groups of Q_6 are $H^0(Q_6,\mathbb{Z}_2)=\mathbb{Z}_2$, $H^1(Q_6,\mathbb{Z}_2)=(\mathbb{Z}_2)^{15}$ and $H^2(Q_6,\mathbb{Z}_2)=(\mathbb{Z}_2)^{14}$.*

The computation that proves this statement occupies the rest of this section.

Proof. The space M_4 is a one-point union of 7 circles, and M_5 of 31 circles, so the only degree where the connecting map d is nontrivial is $i = 1$. It is the sum of 5 maps

$$d_{(j)} : H^1(M_5, \mathbb{Z}_2) \to H^1(M_{4,j}, \mathbb{Z}_2)$$

and

$$\ker d = \cap \ker d_{(j)}.$$

One may compute the dimensions of these kernels using the explicit bases for the first cohomology, that consist of essential crossing types. It is easy to describe $\ker d_{(1)}$ since $d_{(1)}$ maps each essential crossing type either to zero or to another essential crossing type. The space $\ker d_{(1)}$ is defined by 7 linearly independent equations (4) below.

In order to find the equations for $\ker d_{(j)}$ for $j > 1$ we make use of the fact that the action of the symmetric group S_5 on M_5 interchanges these spaces. So we obtain the equations for $\ker d_{(j)}$ from those for $\ker d_{(j-1)}$ by applying the permutation $(j-1\, j)$. The action of the symmetric group may send essential crossing types to non-essential crossing types and the laborious part of the computation is to describe this action. Direct calculation shows that

$$\begin{aligned}
&\dim(\ker d_1) = 24,\\
&\dim(\ker d_1 \cap \ker d_2) = 18,\\
&\dim(\ker d_1 \cap \ker d_2 \cap \ker d_3) = 13,\\
&\dim(\ker d_1 \cap \ker d_2 \cap \ker d_3 \cap \ker d_4)\\
&\qquad\qquad = \dim(\ker d_1 \cap \ker d_2 \cap \ker d_3 \cap \ker d_4 \cap \ker d_5) = 10.
\end{aligned}$$

Therefore,

$$\dim H^1(Q_6, \mathbb{Z}_2) = \dim \ker d + 5 \dim H^0(M_4, \mathbb{Z}_2) = 10 + 5 = 15.$$

From the exact sequence of the previous subsection we also see that

$$\dim H^2(Q_6, \mathbb{Z}_2) = 5 \dim H^1(M_4, \mathbb{Z}_2) - \dim \mathrm{im}(d) = 5 \cdot 7 - (31 - 10) = 14.$$

Now, we provide the details of the computation. We will write a crossing type (I) by listing first the elements of I_0, then, in parentheses, the elements of I_1 and then the elements of I_2. So, for instance, 1(23)45 denotes the crossing type with $I_0 = \{1\}$, $I_1 = \{2, 3\}$ and $I_2 = \{4, 5\}$ and (12)345 is the crossing type with $I_0 = \emptyset$, $I_1 = \{1, 2\}$ and $I_2 = \{3, 4, 5\}$.

The kernel of the map $d_{(1)}$ is spanned by the elements of two types: the crossing types (I) such that $1 \in I_1$ and sums of the following form:

$$\begin{array}{lll} 1(23)45 + (23)145 & 1(24)35 + (24)135 & 12(34)5 + 2(34)15 \\ 14(23)5 + 4(23)15 & 15(23)4 + 5(23)14 & 1(34)25 + (34)125 \\ 13(24)5 + 3(24)15 & & \end{array}$$

The space of equations satisfied by $\ker d_{(1)}$ is the dual subspace to $\ker d_{(1)}$ in $(H^1(M_5))^*$; it is spanned by the elements

$$\begin{array}{lll} 1(23)45^* + (23)145^* & 1(24)35^* + (24)135^* & 12(34)5^* + 2(34)15^* \\ 14(23)5^* + 4(23)15^* & 15(23)4^* + 5(23)14^* & 1(34)25^* + (34)125^* \\ 13(24)5^* + 3(24)15^* & & \end{array} \tag{4}$$

where $(I)^*$ is dual to the crossing type (I). Interchanging the symbols 1 and 2 in an essential crossing type we obtain another essential crossing type. Therefore, the dual to the kernel of $d_{(2)}$ is spanned by

$$\begin{array}{lll} (13)245^* + 2(13)45^* & (14)235^* + 2(14)35^* & 1(34)25^* + 12(34)5^* \\ 4(13)25^* + 24(13)5^* & 5(13)24^* + 25(13)4^* & (34)125^* + 2(34)15^* \\ 3(14)25^* + 23(14)5^* & & \end{array}$$

The permutation $(2\,3)$ sends the kernel of $d_{(2)}$ to the kernel of $d_{(3)}$. The essential crossing types are sent to essential crossing types with one exception: the crossing type 45(12)3 is sent to

$$45(13)2 = 45(12)3 + 4(12)35 + 4(13)25 + 5(12)34 + 5(13)24 + (12)345 + (13)245.$$

The space $(\ker d_{(3)})^*$ is spanned by

$$\begin{array}{ll} (12)345^* + 3(12)45^* + 45(12)3^* & (14)235^* + 3(14)25^* \\ 4(12)35^* + 34(12)5^* + 45(12)3^* & 2(14)35^* + 23(14)5^* \\ 5(12)34^* + 35(12)4^* + 45(12)3^* & (24)135^* + 3(24)15^* \\ & 1(24)35^* + 13(24)5^*. \end{array}$$

Now, consider $\ker d_{(4)}$. There are 4 essential crossing types that are sent to non-essential crossing types by the permutation $(3\,4)$: 5(13)24, 25(13)4, 5(23)14 and 15(23)4. They are sent to the following types:

$$\begin{array}{lcl} 5(13)24 & \mapsto & 5(14)23 = (12)345 + (14)235 + (13)245 + 5(12)34 + 5(13)24, \\ 25(13)4 & \mapsto & 25(14)3 = (12)345 + 5(12)34 + 2(13)45 + 2(14)35 + 25(13)4, \\ 5(23)14 & \mapsto & 5(24)13 = (12)345 + (24)135 + (23)145 + 5(12)34 + 5(23)14, \\ 15(23)4 & \mapsto & 45(23)1 = (12)345 + (23)145 + 4(12)35 + 4(23)15 + 5(12)34 \\ & & \qquad +5(23)14 + 45(12)3, \end{array}$$

respectively. We see that $(\ker d_{(4)})^*$ is spanned by

$$4(23)15^* + (23)145^* + 5(12)34^* + 35(12)4^* + 45(12)3^* + 5(13)24^* + 25(13)4^* + 15(23)4^*$$
$$4(12)35^* + (12)345^* + 5(12)34^* + 45(12)3^*$$
$$14(23)5^* + 1(23)45^* + 15(23)4^*$$
$$34(12)5^* + 3(12)45^* + 35(12)4^*$$
$$5(23)14^* + 5(12)34^* + 35(12)4^* + 45(12)3^* + 5(13)24^* + 25(13)4^* + 15(23)4^*$$
$$4(13)25^* + (13)245^* + 5(13)24^*$$
$$24(13)5^* + 2(13)45^* + 25(13)4^*$$

Finally, let us compute $(\ker d_{(5)})^*$. There are 12 essential crossing types that are sent to non-essential crossing types by the permutation $(4\,5)$:

$$\begin{aligned}
(14)235 &\mapsto (15)234 = (12)345 + (13)245 + (14)235,\\
2(14)35 &\mapsto 2(15)34 = (12)345 + 2(13)45 + 2(14)35,\\
3(14)25 &\mapsto 3(15)24 = (13)245 + 3(12)45 + 3(14)25,\\
23(14)5 &\mapsto 23(15)4 = 2(13)45 + 3(12)45 + 13(14)5,\\
(24)135 &\mapsto (25)134 = (12)345 + (23)145 + (24)135,\\
1(24)35 &\mapsto 1(25)34 = (12)345 + 1(23)45 + 1(24)35,\\
3(24)15 &\mapsto 3(25)14 = (23)145 + 3(12)45 + 3(24)15,\\
13(24)5 &\mapsto 13(25)4 = 1(23)45 + 3(12)45 + 13(24)5,\\
(34)125 &\mapsto (35)124 = (13)245 + (23)145 + (34)125,\\
1(34)25 &\mapsto 1(35)24 = (13)245 + 1(23)45 + 1(34)25,\\
2(34)15 &\mapsto 2(35)14 = (23)145 + 2(13)45 + 2(34)15,\\
12(34)5 &\mapsto 12(35)4 = 1(23)45 + 2(13)45 + 12(34)5.
\end{aligned}$$

As a consequence, $(\ker d_{(5)})^*$ is spanned by

$$(23)145^* + (24)135^* + (34)125^* + 2(34)15^* + 3(24)15^* + 4(12)35^* + 4(13)25^* + 5(23)14^* + 14(23)5^* + 24(13)5^* + 34(12)5^* + 45(12)3^*,$$
$$(12)345^* + (14)235^* + (24)135^* + 1(24)35^* + 2(14)35^* + 4(12)35^* + 5(12)34^* + 45(12)3^*,$$
$$1(23)45^* + 1(24)35^* + 1(34)25^* + 13(24)5^* + 12(34)5^* + 14(23)5^* + 15(23)4^*,$$
$$3(12)45^* + 3(14)25^* + 3(24)15^* + 13(24)5^* + 23(14)5^* + 34(12)5^* + 35(12)4^*,$$
$$4(23)15^* + 4(12)35^* + 34(12)5^* + 45(12)3^* + 4(13)25^* + 24(13)5^* + 14(23)5^*,$$
$$(13)245^* + (14)235^* + (34)125^* + 1(34)25^* + 3(14)25^* + 4(13)25^* + 5(13)24^*,$$
$$2(13)45^* + 2(14)35^* + 2(34)15^* + 12(34)5^* + 23(14)5^* + 24(13)5^* + 25(13)4^*.$$

Computation show that there are 21 linearly independent equations in 31 variables, which gives the dimension of $\ker d_1 \cap \ker d_2 \cap \ker d_3 \cap \ker d_4 \cap \ker d_5$ equal to 10. $\square$

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