Morfismos, Vol. 24, No. 2, 2020, pp. 27–33

Sequential motion planning in polyhedral products of connected sums of real projective spaces

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Abstract

In this short note we compute the higher topological complexity of polyhedral products constructed out of connected sums of real projective spaces. We show that the answer is maximal possible. In particular, the higher topological complexity of polyhedral products whose factors are closed (orientable or not) surfaces will be calculated as well.

2010 Mathematics Subject Classification: Primary 55S40, 55M30; Secondary 70Q05.
Keywords and phrases: higher topological complexity, polyhedral product, connected sum of real projective spaces.

1 Introduction

The topological complexity of a path connected space X, $\mathrm{TC}(X)$, is a homotopy invariant of X introduced by Farber in [6] in order to pave the way for topological aspects of the motion planning problem in robotics. Then, Rudyak defined in [8] a new family of homotopy invariants $\mathrm{TC}_r(X)$ of X, for $r \geq 2$, with the key property that $\mathrm{TC}_2(X)$ turns out to be $\mathrm{TC}(X)$. These numerical invariants are called sequential or higher topological complexities of X.

In the case of a g-iterated connected sum of $\mathbb{R}\mathrm{P}^n$ with itself $g\mathbb{R}\mathrm{P}^n$ ($g, n \geq 2$), it was shown in [1, Theorem 1.1] that, for $r \geq 3$, $\mathrm{TC}_r(g\mathbb{R}\mathrm{P}^n)$ achieves its maximal possible value by considering a simple zero-divisor cup-length argument, this is, $\mathrm{TC}_r(g\mathbb{R}\mathrm{P}^n) = rn$. Such a fact contrasts with the sophisticated machinery used in [5] to prove that $\mathrm{TC}(g\mathbb{R}\mathrm{P}^n) = 2n$.

In this short note we exhibit that the higher topological complexities of polyhedral products constructed out of connected sums of real projective spaces reach their maximal possible value as well (Proposition 3.2 below). As an immediate consequence, we will see that the same feature holds for polyhedral products whose factors are closed (orientable or not) surfaces (Section 3.2).

2 Preliminaries

2.1 Topological complexity

For $r \geq 2$, the rth topological complexity of a path-connected space X, denoted $\mathrm{TC}_r(X)$, is defined to be the least nonnegative integer n for which there is an open cover $\{U_0, \ldots, U_n\}$ of X^r, the cartesian product of r copies of X, on each on which there exits a continuous section $s_i : U_i \to X^{[0,1]}$ of the fibration

$$\begin{array}{rccl} e_r\colon & X^{[0,1]} & \to & X^r \\ & \gamma & \mapsto & \left(\gamma\left(\frac{0}{r-1}\right), \gamma\left(\frac{1}{r-1}\right), \ldots, \gamma\left(\frac{r-1}{r-1}\right)\right). \end{array}$$

The collection $\{\mathrm{TC}_r(X)\}_{r\geq 2}$ is called the higher or sequential topological complexities of X, while $\mathrm{TC}(X) := \mathrm{TC}_2(X)$ is known as the *topological complexity* of X.

A close relative of $\mathrm{TC}(X)$ is the LS category of X. The latter, denoted $\mathrm{cat}(X)$, is defined as the smallest n for which there exists an open cover $\{U_0, \ldots, U_n\}$ of X with each U_i being contractible *within* X. The following result describes the main connection between $\mathrm{TC}_r(X)$ and $\mathrm{cat}(X)$, both homotopy invariants of X.

Theorem 2.1. *[3, Theorem 3.9] For a path-connected space X having the homotopy type of a* CW *complex,*

$$\mathrm{cl}(X) \leq \mathrm{cat}(X) \leq \mathrm{hdim}(X) \quad \textit{and} \quad \mathrm{zcl}_r(X) \leq \mathrm{TC}_r(X) \leq r\,\mathrm{cat}(X),$$

where $\mathrm{hdim}(X)$ *denotes the (cellular) homotopy dimension of X, i.e., the smallest dimension of* CW *complexes having the homotopy type of X.*

The notation $\mathrm{cl}(X)$ means the cup-length of X, which is defined to be the greatest integer n for which there exist n positive dimensional cohomology classes $\xi_i \in \tilde{H}^*(X; R)$, with R being a commutative ring with unit, such that $\xi_1 \cdots \xi_n \neq 0$. On the other hand, the rth zero-divisor cup-length of X, denoted $\mathrm{zcl}_r(X)$, is the length of the longest nontrivial product in $\ker(\Delta_r^*\colon H^*(X^r;R) \to H^*(X;R))$, where $\Delta_r\colon X \to X^r$ is the r-fold iterated diagonal.

2.2 Polyhedral products

Let $(\underline{X},\underline{A}) = \{(X_i,A_i)\}_{i=1}^m$ be a family of pairs of spaces and K be an abstract simplicial complex on vertices $\{1,\ldots,m\}$. The polyhedral product determined by $(\underline{X},\underline{A})$ and K is defined to be[1]

$$(\underline{X},\underline{A})^K = \cup_{\sigma\in K}(\underline{X},\underline{A})^\sigma, \tag{1}$$

where $(\underline{X},\underline{A})^\sigma = \prod_{i=1}^m Y_i$ and

$$Y_i = \begin{cases} A_i, & \text{if } i \in \{1,\ldots,m\}\setminus\sigma; \\ X_i, & \text{if } i \in \sigma. \end{cases}$$

Throughout this paper we are only interested in the case where all $A_i = *$. Thus, $(\underline{X},*)^K$ and $(\underline{X},*)^\sigma$ are simply denoted by $\underline{X}^K$ and $\underline{X}^\sigma$, respectively. Moreover, it is clear that, for any $\sigma \in K$, $\underline{X}^\sigma$ is a retract of $\underline{X}^K$ and $\underline{X}^\sigma \approx \prod_{i\in\sigma} X_i$.

Example 2.2. Let $\underline{X} = \{(X_i,*)\}_{i=1}^m$ be a family of based spaces. If K denotes the standard $(m-1)$-simplex Δ^{m-1}, then $\underline{X}^K$ turns out to be the cartesian product $X_1\times\cdots\times X_m$. On the other hand, if K is zero dimensional, this is, $K=\{\{1\},\{2\},\ldots,\{m\}\}$, then $\underline{X}^K = \vee_{i=1}^m X_i$.

Example 2.3. Let $\underline{X} = \{(X_i,*)\}_{i=1}^3$ be a family of based spaces. The simplicial complex K with facets $\{1,2\}$ and $\{3\}$ gives rise to the polyhedral product $\underline{X}^K = (X_1\times X_2)\vee X_3$, while

$$\underline{X}^K = (X_1\times X_2\times *)\cup(X_1\times *\times X_3)\cup(*\times X_2\times X_3)$$

is obtained by taking K to be the boundary of the standard 2-simplex Δ^2.

In order to compute the higher topological complexities of polyhedral products whose factors are connected sums of real projective spaces, it will be convenient to take into account the main properties related to cat and TC_r of general polyhedral products that were widely studied in [2]. Before making explicit such results, we need:

[1] Note that $(\underline{X},\underline{A})^{\sigma_1}$ is contained in $(\underline{X},\underline{A})^{\sigma_2}$ provided $\sigma_1\subset\sigma_2$. Therefore, it suffices to take the union over all the maximal simplices of K in (1), this is, simplices that are not contained in any other simplex of K.

Definition 2.4. A family of based spaces $\underline{X} = \{(X_i, *)\}_{i=1}^m$ is said to be LS-logarithmic if

$$\operatorname{cat}(X_{i_1} \times \cdots \times X_{i_k}) = \operatorname{cat}(X_{i_1}) + \cdots + \operatorname{cat}(X_{i_k})$$

holds provided $i_1 < \cdots < i_k$. Likewise, $\underline{X}$ is said to be TC_r-logarithmic, for some $r \geq 2$, if

$$\mathrm{TC}_r(X_{i_1} \times \cdots \times X_{i_k}) = \mathrm{TC}_r(X_{i_1}) + \cdots + \mathrm{TC}_r(X_{i_k})$$

holds provided $i_1 < \cdots < i_k$.

Theorem 2.5. *[2, Lemma 6.7, Theorem 1.4] Let $\underline{X}^K$ be the polyhedral product associated to a family of based spaces $\underline{X} = \{(X_i, *)\}_{i=1}^m$ and an abstract simplicial complex K. We have*

$$\operatorname{cat}(\underline{X}^K) \leq \max\left\{\operatorname{cat}(X_{i_1}) + \cdots + \operatorname{cat}(X_{i_n}) : \{i_1, \ldots, i_n\} \in K\right\}.$$

Further, if the family $\{(X_i, *)\}_{i=1}^m$ is LS*-logarithmic, then the latter inequality is in fact an equality.*

Theorem 2.6. *[2, Theorem 1.5] Let $\{(X_i, *)\}_{i=1}^m$ be a collection of based spaces. If, for some $r \geq 2$,*

1. $\mathrm{TC}_r(X_i) = r\operatorname{cat}(X_i)$ *for all* $i \in \{1, \ldots, m\}$,
2. *the collection $\{(X_i, *)\}_{i=1}^m$ is* LS*-logarithmic, and*
3. *the collection $\{(X_i, *)\}_{i=1}^m$ is TC_r-logarithmic,*

then

$$\mathrm{TC}_r(\underline{X}^K) = r\operatorname{cat}(\underline{X}^K) = \max\left\{\sum_{i\in\sigma} \mathrm{TC}_r(X_i) : \sigma \in K\right\}.$$

3 Main results

Throughout this section $\underline{gP}^K$ denotes the polyhedral product associated to the based family $\underline{gP} = \{(g_i\mathbb{R}\mathrm{P}^{n_i}, *)\}_{i=1}^m$ and an abstract simplicial complex K, with $g_i, n_i \geq 2$.

3.1 The higher topological complexities of $\underline{gP}^K$

Before delving into the computation of $\mathrm{TC}_r(\underline{gP}^K)$ for $r \geq 3$, it will be necessary to bear in mind the mod-2 cohomology ring structure of each polyhedral product factor. Explicitly, it was shown in [1, Lemma 2.1] that the mod-2 cohomology ring of $g\mathbb{R}\mathrm{P}^n$, $g, n \geq 2$, is generated by 1-dimensional cohomology classes x_q, with $1 \leq q \leq g$, subject to the relations:

- $x_q x_\ell = 0$ for $q \neq \ell$;
- $x_q^{n+1} = 0$;
- $x_q^n = x_\ell^n$.

Consequently, the top class in $H^*(g\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ is given by any power x_q^n, with $1 \leq q \leq g$, and it is simply denoted by t.

Proposition 3.1. *The family $\underline{gP}$ is* LS*-logarithmic.*

Proof. It suffices to show that

$$\mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2}) = n_1 + n_2 = \mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1}) + \mathrm{cat}(g_2\mathbb{R}\mathrm{P}^{n_2}).$$

Furthermore, since

$$\mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2}) \leq \mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1}) + \mathrm{cat}(g_2\mathbb{R}\mathrm{P}^{n_2}) \leq n_1 + n_2,$$

it remains to prove that $\mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2})$ is bounded from below by $n_1 + n_2$. This easily follows from the Künneth formula, because $H^{n_1+n_2}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2}; \mathbb{Z}_2)$ is generated by $t_1 \otimes t_2$, where t_i is the top class in $H^*(g_i\mathbb{R}\mathrm{P}^{n_i}; \mathbb{Z}_2)$. Hence, by Theorem 2.1, $n_1 + n_2 \leq \mathrm{cl}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2}) \leq \mathrm{cat}(g_1\mathbb{R}\mathrm{P}^{n_1} \times g_2\mathbb{R}\mathrm{P}^{n_2})$. ☐

Proposition 3.2. *For $r \geq 3$, the rth topological complexity of $\underline{gP}^K$ is given by*

$$\mathrm{TC}_r(\underline{gP}^K) = r\,\mathrm{cat}(\underline{gP}^K) = \max\left\{\sum_{i\in\sigma} \mathrm{TC}_r(g_i\mathbb{R}\mathrm{P}^{n_i}) \colon \sigma \in K\right\}.$$

Proof. We only need to check that the hypotheses of Theorem 2.6 hold. The first one follows from [1, Theorem 1.1] because $\mathrm{TC}_r(g_i\mathbb{R}\mathrm{P}^{n_i}) = rn_i = r\,\mathrm{cat}(g_i\mathbb{R}\mathrm{P}^{n_i}) = \mathrm{zcl}_r(g_i\mathbb{R}\mathrm{P}^{n_i})$, with $r \geq 3$ and $g_i, n_i \geq 2$.

Hence, in view of Proposition 3.1, it remains to prove that the family $\underline{gP}$ is TC_r-logarithmic. This fact comes from a standard argument: note that

$$r\left(\sum_{j=1}^{k} n_{i_j}\right) = \sum_{j=1}^{k} \mathrm{zcl}_r(g_i\mathbb{R}\mathrm{P}^{n_{i_j}}) \leq \mathrm{zcl}_r\left(\prod_{j=1}^{k} g_i\mathbb{R}\mathrm{P}^{n_{i_j}}\right)$$
$$\leq \mathrm{TC}_r\left(\prod_{j=1}^{k} g_i\mathbb{R}\mathrm{P}^{n_{i_j}}\right) \leq r\left(\sum_{j=1}^{k} n_{i_j}\right),$$

where the first inequality follows from [4, Lemma 2.1], the second and the third ones come from Theorem 2.1. This completes the proof. □

3.2 Closed surfaces

The closed non-orientable surface N_g of genus $g \geq 1$ is obtained by taking the g-iterated connected sum of $\mathbb{R}\mathrm{P}^2$ with itself. So, if $\underline{N} = \{(N_{g_i}, *)\}_{i=1}^m$ denotes the corresponding LS-logarithmic family of based spaces, then one can easily compute $\mathrm{TC}_r(\underline{N}^K)$, for $r \geq 3$, in view of Theorem 2.6. The fundamental equalities $\mathrm{TC}_r(N_{g_i}) = r\,\mathrm{cat}(N_{g_i}) = \mathrm{zcl}_r(N_{g_i}) = 2r$ (mod-2 coefficients), for $g_i \geq 1$ and $r \geq 3$, were originally proved in [7, Proposition 5.1]. The TC_r-logarithmicity hypothesis of $\underline{N}$ follows from similar ideas to those shown in the proof of Proposition 3.2. Hence, for $r \geq 3$,

$$\mathrm{TC}_r(\underline{N}^K) = r\,\mathrm{cat}(\underline{N}^K) = 2r(1 + \dim K).$$

On the other hand, the orientable case is identical to the previous one. Concretely, if $\underline{\Sigma} = \{(\Sigma_{g_i}, *)\}_{i=1}^m$ stands for a family of closed orientable surfaces Σ_{g_i} of genus $g_i \geq 2$, then the family $\underline{\Sigma}$ turns out to be LS-logarithmic. The crucial ingredient $\mathrm{TC}_r(\Sigma_{g_i}) = r\,\mathrm{cat}(\Sigma_{g_i}) = \mathrm{zcl}_r(\Sigma_{g_i}) = 2r$, for $r \geq 3$, originally comes from [7, Proposition 5.1] as well (here, the coefficients are taken in $\mathbb{Q}$). In conclusion, using Theorem 2.6, we obtain that, for $r \geq 3$, $\mathrm{TC}_r(\underline{\Sigma}^K)$ attains its maximal possible value, i.e.,

$$\mathrm{TC}_r(\underline{\Sigma}^K) = r\,\mathrm{cat}(\underline{\Sigma}^K) = 2r(1 + \dim K).$$

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