Morfismos, Vol. 22, No. 2, 2018, pp. 121–173

Sheaf of categories and categorical Donaldson theory

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Abstract

In this paper we take a new look at categorical linear systems applying the technique of sheaves of categories. We combine this technique with the theory of categorical Kähler metrics in order to build two parallels:

1) A parallel with Donaldson theory of Kähler-Einstein metrics.

2) A parallel with Donaldson theory of polynomial invariants.

As an outcome we introduce sheaves of categories which cannot be connected to potentials and obstructions to that are the moduli spaces of stable objects. Connections of sheaves of categories with Homological Mirror Symmetry for non-complete intersections and the procedure of arborealization are discussed as well.

2010 Mathematics Subject Classification: 14A22, 14J33, 53D37.
Keywords and phrases: Sheafs of categories, Donaldson theory.

1 Introduction

The theory of linear systems is 2000 years old. Recently a new read of this theory was suggested by the authors in [18]. Based on the recent breakthrough made by Haiden, Katzarkov, Kontsevich, Pandit [15], who introduced the theory of categorical Kähler metrics, we develop further the theory of categorical linear systems.

In this paper we take a new look at the categorical linear systems applying the technique of sheaves of categories (see section 3). We combine this technique with the theory of categorical Kähler metrics in order to build two parallels:

1. A parallel with Donaldson theory of Kähler-Einstein metrics - [9].

2. A parallel with Donaldson theory of polynomial invariants - [8].

We briefly recall the theory of categorical Kähler metrics developed in [15]. We represent this work in terms of a categorical Donaldson-Uhlenbeck-Yau correspondence outlined in the table below. We start with the classical GIT theory of the action of a group G with a unitary subgroup K on a manifold X.

$$\begin{tikzcd} X/K \arrow[r] & X/G \\ \Phi^{-1}(0) \arrow[u, hook] \arrow[ur] & \text{Kempf-Ness Functional} \end{tikzcd}$$

The categorical interpretation of the above classical GIT setup, as followed in [15], can be presented as follows:

$$\begin{tikzcd} \mathcal{C}^0 \arrow[rr] & & \mathcal{C} \quad \text{— } S_{\mathbb{C}} \\ & \mathcal{D} \arrow[ul] \arrow[ur] \quad \text{— stable objects, stable Lagrangians} & \end{tikzcd}$$

The theory developed in [15] is based on the following correspondences:

Table 1

$\mathcal{C}$	$X/G \curvearrowleft G$
$\mathcal{C}^0$	X/K
equivariant line bundle	$\Phi : X \to K^V$
$Met(E)$	G/K
Flow	pullback of grat Φ to G/K
Mass M	Φ
$M < \lvert Z\rvert$	Bogomolny condition

We briefly explain the above categorical setup in section 5.

In his seminal work [7] Donaldson used the non-rationality of Dolgachev surfaces in order to find examples of homeomorphic but non-diffeomorphic surfaces. After developing the theory of sheaves of categories we propose a categorical interpretation of Donaldson's results.

We start with replacing the classical blow-up formulas in Donaldson theory with a blow-up formula of semi-orthogonal decompositions.

Conjecture 1.1. Let $C =< C_1, C_2 >$ be a semi-orthogonal decomposition of a triangulated category C by admissible subcategories C_1, C_2. (Typical examples we have in mind are the bounded derived categories of coherent sheaves and Fukaya-Seidel categories - see e.g. [2].) Let Φ be a categorical version of Donaldson polynomial invariants (see section 7). Then

$$\Phi(C) = \Phi(C_1) \cdot \Phi(C_2) \cdot T(C), \tag{1}$$

where $T(C)$ is a standard term.

In section 7 we discuss wall crossing issues connected with the above conjecture. We consider some far-going applications of this conjecture in section 7. Using the categorical Kähler metrics we define canonical categorical Kähler-Einstein degenerations. We formulate:

Conjecture 1.2. Let $X_0 \cup X_i$ be a canonical categorical Kähler-Einstein degeneration. Then we have the following equality defining the soul of polynomial invariants:

$$(X_0 \cup X_i) = \Phi(X_0) \cdot T(C). \tag{2}$$

A major building block for applications is the following:

Conjecture 1.3. Let LG be a part of a four-dimensional LG model which contains all vanishing cycles corresponding to all cohomologies but $h^{0,0}$, $h^{4,4}$. Assume that these vanishing cycles produce $b_2^+ \geq 1$ cohomologies of a minimal symplectic fourfold. Then $LG \neq \mathrm{LG}_1 \# \mathrm{LG}_2$, where each LG_i is a LG model with $b_2^+ \geq 1$ mirrors of minimal symplectic fourfolds.

At the moment the conjectures formulated above are rather vague. In section 7 we outline some possible applications in the case of Fukaya categories. The considerations above suggest that we have a parallel:

Classical	Categorical
Donaldson invariants	categorical invariants
non-diffeomorphic manifolds	non-rational varieties

This parallel suggests that categorical Donaldson polynomial invariants are very deep invariants which measure the gluing of categories. In section 7 we spell out the connection between categorical basic classes and gaps in Orlov spectra as well as the connection between basic classes and dynamical spectra of the corresponding Fano manifolds.

We propose that categorical Donaldson invariants define new birational invariants and we develop two techniques for studying the theory of categorical Donaldson invariants - categorical linear systems and the theory of sheaves of categories. There might be better ways of doing that. We discuss different applications of the idea of shaves of categories related to the proof of Homological Mirror Symmetry (HMS) for non-complete intersections. An important outcome of this paper is the following correspondence.

Classical Donaldson Theory	Categorical Donaldson Theory
$X \neq X_1 \#_C X_2$	sheaf of categories is not connected with a potential
nontrivial moduli spaces	nontrivial moduli spaces of stable objects
wall crossing on metrics	wall crossing recorded in sheaves of categories

The paper is organized as follows. In section 2 we recall the theory of categorical linear systems. In section 3 we develop the theory of sheaves of categories. In section 4 we introduce the notion of categorical Kähler metrics. In section 5 we build the theory of categorical Okounkov bodies and in section 6 categorical Kähler-Einstein metrics. In section 7 we consider further applications.

2 Categorical linear systems

In this section we define two new notions - categorical linear systems and categorical base loci. We try to indicate the potential of these new notions for studying categories by connecting them with well-known categorical notions - gaps of spectra and phantoms.

Let T be a saturated dg-category. Consider the endofunctors A, F of T.

Definition 2.1. A **noncommutative linear system** is a collection of morphisms $s \in \mathrm{Hom}(A, F)$. A pair of morphisms, $s_1, s_2 \in \mathrm{Hom}(A, F)$, is a **noncommutative pencil**.

We may think of these morphisms as natural transformations $s_i : A \to F$. We also define:

Definition 2.2. The **scheme-theoretic base locus** of a noncommutative linear system $\mathcal{C}$ is the full subcategory of objects of T on which all $s \in \mathcal{C}$ vanish in the homotopy category. The **triangulated base locus** of $\mathcal{C}$ is the full subcategory of objects of T on which all $s \in \mathcal{C}$ act nilpotently in the homotopy category.

Consider the case where X is an algebraic variety, $T = \mathrm{D}^{\mathrm{b}}(X)$ and F corresponds to tensoring with a line bundle. In this case, a noncommutative linear system abuts to the classical notion of a linear system by taking the homotopy classes of these morphisms. The scheme-theoretic base locus is precisely the full subcategory of complexes such that the cohomology is scheme-theoretically supported on the base locus. This is not a triangulated category. On the other hand, the triangulated base locus is the full subcategory of objects on which the cohomology is set-theoretically supported on the base locus. This is a triangulated category.

Now, let L be any object of T. Consider

$$R_{L,F} = \bigoplus_{n=0} \mathrm{RHom}(L, F^n(L)).$$

We consider every r in $\mathrm{RHom}(L, F^n(L))$, $n > 0$ as a morphism of graded bimodules $r : \mathrm{RHom}(L, F^n(L)) \to \mathrm{RHom}(L, F^n(L))[i]$. We define $\mathrm{Tors}(R_{L,F})$ to be the full subcategory consisting of all objects T in grmod over $R_{L,F}$ such that for every r in R_L there exists $N >> 0$ such that $r^N(T) = 0$. Finally we define $\mathrm{DGProj}(R_{L,F}) =$ grmod over $R_{L,F}/\mathrm{Tors}(R_{L,F})$.

Definition 2.3. We define $\mathrm{Tors}(R_{L,F})$ to be the L-**base locus** of the functor F.

The definition above is very complex and suggests that the categorical base locus measures the complexity of the functor F for a reasonable choice of the object L.

In the case where F is a twist by a very ample line bundle on a smooth projective variety X and L is a line object (see e.g. [10]) in $\mathrm{D}^{\mathrm{b}}(X)$, we get that $\mathrm{DGProj}(R_{L,F}) =$ grmod over $R_{L,F}/\mathrm{Tors}(R_{L,F})$ is just $\mathrm{D}^{\mathrm{b}}(X)$. For Artin-Zhang twists we get some noncommutative deformations of $\mathrm{D}^{\mathrm{b}}(X)$. But for more general functors some new phenomena appear in this categorical setting.

The approach we suggest records the categorical base loci via marking divisors at infinity of the LG model, where we take the point of view that functors with high exts lead to bigger gaps in the Orlov spectra. In some cases the last phenomenon is recorded by the monodromy of the LG model. When working with mixed variations of stability structures we note that nontrivial exts in $\mathrm{DGProj}(R_{L,F}) =$ grmod over $R_{L,F}/\operatorname{Tors}(R_{L,F})$ are connected with ghost sequences of length equal to the nontrivial exts.

The categories $\mathrm{DGProj}(R_{L,F}) =$ grmod over $R_{L,F}/\operatorname{Tors}(R_{L,F})$ and $\operatorname{Tors}(R_{L,F})$ behave well under the following operations:

1. Birational maps.

2. Taking invariant or anti-invariant parts or combining F with any Schur functor.

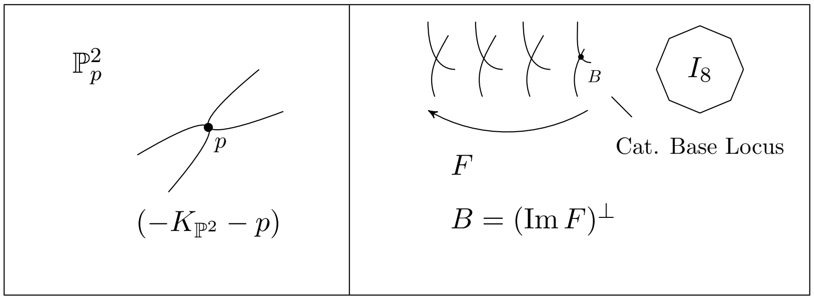
3. We can modify $\operatorname{Tors}(R_{L,F})$ to be defined as a full subcategory consisting of all objects T such that for every r in $R_{L,F}$ there exists $N < k_i$ such that $r^N(T) = 0$. So we have $\operatorname{Tors}(R_{L,F})_{k_1} \subset \operatorname{Tors}(R_{L,F})_{k_2}$ for $k_1 < k_2$.

4. Pencils, nets as well as fibrations of categories can be defined by choosing sections in
$$\mathrm{DGProj}(R_{L,F}) = \text{grmod over } R_{L,F}/\operatorname{Tors}(R_{L,F}).$$
Using the ghost sequences of the base categories ($\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$, $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2)$ and so on) we obtain ghost sequences for
$$\mathrm{DGProj}(R_{L,F}) = \text{grmod over } R_{L,F}/\operatorname{Tors}(R_{L,F}).$$

5. Assume that the functor F splits as a product of functors $F = F_m \cdots F_1$. then we have $R_{L,F} \subset R_{L,F_m} \cdots R_{L,F_1}$. This formula will be implemented as the main ingredient of the K-calculus. Both formulas provide us with the opportunity to "glue" ghost sequences in order to calculate Orlov spectra.

We give some simple examples.

Table 2



1. Consider $\mathbb{P}^2_p$ - the image of $\mathbb{P}^2$ by the anti-canonical system with one base point p. Consider its mirror - an elliptic fibration with 4 fibers with usual double points singularities and an I_8 fiber - see Table 2. The category generated by the image of the thimble vanishing in the fourth singular fiber in the generic open elliptic curve is the categorical base locus for the functor rotation around infinity. This is a simple consequence of Homological Mirror Symmetry - see e.g. [2]. There are two ways we can think of the creation of this base locus:

   1) We localize FS category of the LG model for $\mathbb{P}^2_p$ by one thimble corresponding to the point p. (We return this singular fiber to infinity.)
   2) We mark the point on the circle configuration of rational curves I_9.

   Both of these correspond to creating classical base loci of the linear system $-K_{\mathbb{P}^2} - p$. So by analogy with the classical situation we will think of this base locus as the marking of a point on the fiber at infinity. This marked point (this localized thimble) becomes a base point, categorical base locus, for the functor twist by $-K_{\mathbb{P}^2} - p$ - the rotation around the fiber at infinity with the marked point fixed. We can think of this point as slightly moved from infinity but still close to infinity. The rotation functor keeps it fixed.

2. The example above can be interpreted as a projection functor. In general projection functors produce many examples of categorical liner systems many of which are new non-classical examples. Partial rotations in LG models also provide such examples.

3. Noncommutative Lefschetz pencils:

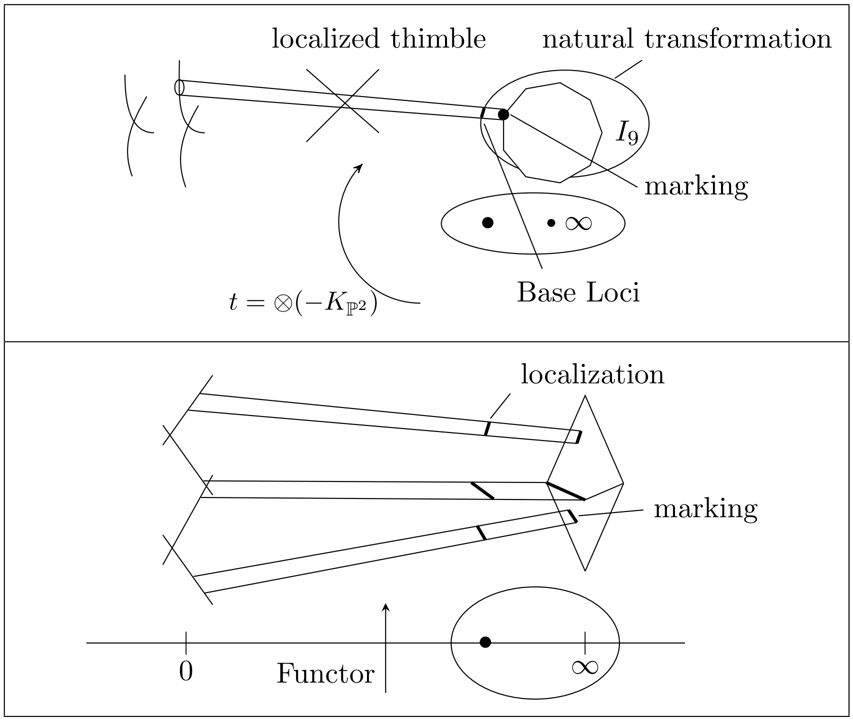
   a) Week notion - two natural transformations s, t from the identity functor to the functor F. This does not give a functor from $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$ to the category T, because there is no requirement that $st = ts$. This notion produces a $\mathbb{P}^1$-family of noncommutative divisors (the linear combinations of s and t) and the notion of base locus (objects on which s and t both vanish).

   b) The stronger notion requires a functor from $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$ to the category T, which amounts to the requirement that all the natural transformations considered must commute with each other.

   c) An intermediate notion which does not require $st = ts$, but only that st and ts agree up to some multiplicative factor - for 3 natural transformations r, s, t we just ask that the 9 natural transformations $r^2, s^2, t^2, rs, st, tr, rt, ts, sr$ satisfy 3 linear relationships so that their span has rank 6 (as in a noncommutative $\mathbb{P}^2$).

By analogy with the classical situation we will call the pencils from c) topological and the pencils from b) algebraic. We will work with algebraic noncommutative systems mainly. For a pictorial explanation of categorical base loci for Fukaya-Seidel categories - see Table 3. We can think of natural transformations of rotation functors and the identity functor as paths around the fiber at infinity. Intersections of these paths are the categorical base loci - the thimbles we have localized by. The geometry of this marked set plays an important role in our considerations.

Table 3: Categorical Base Loci



Blowing up this base locus corresponds to creating a fiber at the LG model - see [2]. We move now to the definition of categorical multiplier ideal sheaf. Classically multiplier ideal sheaf is defined as follows. For a projective variety X and a linear system of the divisor D we define $J_{\lambda_i}(D) = \mu'_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \lambda_i \mu^* \Sigma_i s_i \cdot E_i \rfloor))$, where $s_i \cdot E_i$ are divisors in the exceptional loci. We obtain the classical multiplier ideal sheaf by resolving singularities and taking the floor function, corresponding to taking parts of these divisors. As a result $J_{\lambda_i}(D)$ measure singularities of the pair (X, D).

We define the categorical multiplier ideal sheaf based on the approach via

$$\mathrm{DGProj}(R_{L,F}) = \mathrm{grmod\ \ over}\ R_{L,F}/\operatorname{Tors}(R_{L,F}) \tag{3}$$

developed above. We consider categorically a sequence of functors $\lambda_i F$ acting on modified categories C^i defined as

$$J(C^i, \lambda_i F) = \mathrm{DGProj}(R_{L_i,\lambda_i F}) = \mathrm{grmod\ \ over}\ R_{L_i,\lambda_i F}/\operatorname{Tors}(R_{L_i,\lambda_i F}). \tag{4}$$

If the functor $F = F\lambda_k \cdots F\lambda_1$, then $\lambda_i F = F\lambda_{i-1} \cdots F\lambda_1$.

Definition 2.4. We define the sequence of categories $J(C^i, \lambda_i F)$ to be a **categorical multiplier ideal sheaf**.

In the case of X being a smooth projective variety, L a line object (e.g. $\mathcal{O}_X$) and F a twist by an ample line bundle $R_{L,\lambda_i F}$ we have an analogue to the classical multiplier ideal sheaf situation. Indeed if F is a twist by a divisor $D = D_1 + \cdots + D_k = \Sigma_i s_i \cdot E_i$ we get a functor $F = F_k \cdots F_1$. This observation suggests a generalization - the definition of categorical multiplier ideal sheaf. Assume that F_i commute. We get $R_{L,F} \subset R_{L,F_k \cdots F_1}$ with the corresponding sequence of categories. From this prospective:

$$F = F_k, \quad \lambda_{k-1}F = F_k \cdot F_{k-1}, \quad \ldots, \quad \lambda_1 F = F_k \cdots F_1. \tag{5}$$

So the categorical multiplier ideal sheaf is defined by

$$J(C^i, \lambda_i F) = \mathrm{DGProj}(R_{L_i,\lambda_i F}) = \mathrm{grmod\ \ over}\ R_{L_i,\lambda_i F} / \operatorname{Tors}(R_{L_i,\lambda_i F}) \tag{6}$$

as a sequence of localizations which measure the complexity of the functor F. Classically for the mixed Hodge structure (MHS) associated with the function f defining D we have a spectrum of the MHS (see [5]). The monodromy $e^{i.\lambda_i}$ of the MHS of f is connected with the classical multiplier ideal sheaf $J(X, \lambda_i F)$. By analogy with [5] we conjecture that there exists a matrix factorization category MF so that the spectrum of the mixed noncommutative Hodge structure associated with it produces the jumping numbers of the categorical multiplier ideal sheaf. We give as an example the following theorem - see [18].

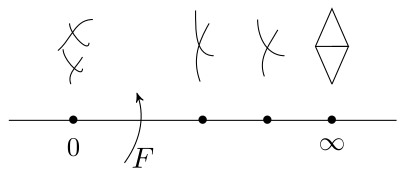
Theorem 2.5. *The multiplier ideal sheaf for the category A_n and the localization functor - restricting to A_{n-1} determine the Orlov spectrum of A_n.*

In this case the categorical multiplier ideal sheaf is a sequence of localizations. We plan to compute more examples. One important case is the so-called LG functor.

Definition 2.6. We will call F a **LG functor** of the Fukaya-Seidel category if F is a functor of a rotation around the fiber at infinity and some other fibers of the LG model associated with the mirror of a smooth projective variety.

This means that these fibers are unchanged under the rotations - see Figure 1.

Figure 1: Canonical LG functors



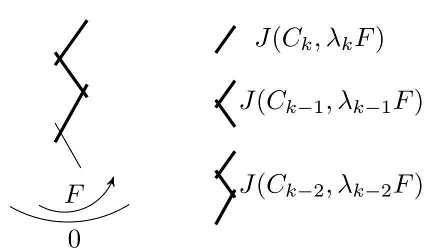
Examples of LG functors are the A side realizations of the Serre functor. In most of the examples in this paper we will consider the mirrors of smooth Fano manifolds. Most of the LG models associated with the mirror of a smooth projective Fano manifold have a singular fiber at infinity (this is certainly true for non-rational Fano manifolds).

Definition 2.7. We will call F a **canonical LG functor** of the Fukaya-Seidel category of the mirror of a smooth Fano manifold if F is a functor of a rotation around the fiber at infinity and all other nonzero singular fibers.

In other words all fibers but the zero fiber is left fixed or this is a rotation around the fiber at zero. Building the theory of categorical multiplier ideal sheaves for LG functors and especially for canonical LG functors is the main part of this project. The categorical multiplier ideal sheaf

$$J(C_k, \lambda_k F) \subset \cdots \subset J(C_1, \lambda_1 F) \tag{7}$$

in the case of LG functors is a sequence of localizations $J(C_i, \lambda_i F)$. By analogy with the A_n case we start with the localization by the biggest thimble and we obtain $J(C_k, \lambda_k F)$ - this corresponds to marking the whole singular fiber over zero. We start unmarking this singular fiber so that the rank $HP(T)$ of the category T we localize by goes down by one.



Similarly we can define the categorical multiplier ideal sheaf

$$J(C_k, \lambda_k F) \subset \cdots \subset J(C_1, \lambda_1 F) \tag{8}$$

for the derived category of singularities. In this case localizations are nothing else but blowing-down components of the central fiber, A deeper analysis of

$$\mathrm{DGProj}(R_{L,F}) = \mathrm{grmod}\ \text{ over } R_{L,F}/\operatorname{Tors}(R_{L,F}) \text{ and } \operatorname{Tors}(R_{L,F})$$

supports these expectations.

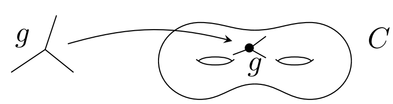
3 Real blow-ups, sheaves of categories and linear systems

In this section we develop a connection between sheaves of categories and LG models. We first define a sheaf of categories. Our treatment is parallel to [17] but somewhat different.

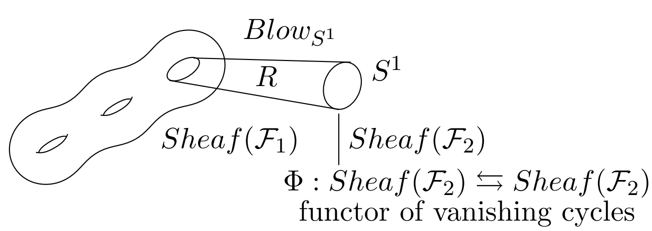
Definition 3.1. Let *Graph* be the incidence category of a graph Γ. Let $\mathcal{F}$ be a topological local system of categories. We call

$$Sheaf(\mathcal{F}) := \varinjlim_{g} Func(C_{graph} \mapsto \mathcal{F}), \qquad g \in Graph \tag{9}$$

a **simple sheaf of categories**, where g is the incidence graph of the embedded in C_{graph}.



Definition 3.2. Let $Sheaf(\mathcal{F}_1)$ and $Sheaf(\mathcal{F}_2)$ be two simple sheaves of categories and $Blow_{S^1}$ be a real blow-up of C.



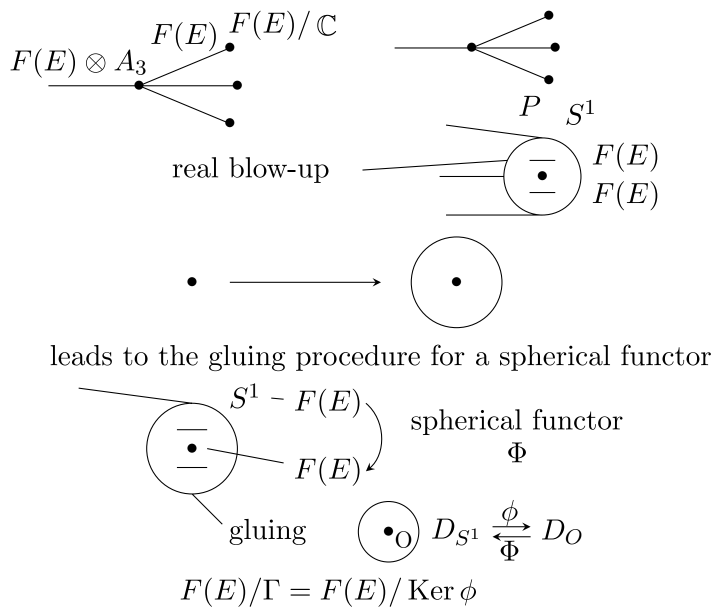
We call $Sheaf(\mathcal{F}_1) \bigcup_R Sheaf(\mathcal{F}_2)$ a **perverse sheaf of categories over** C glued by the functor R.

Example 3.3.

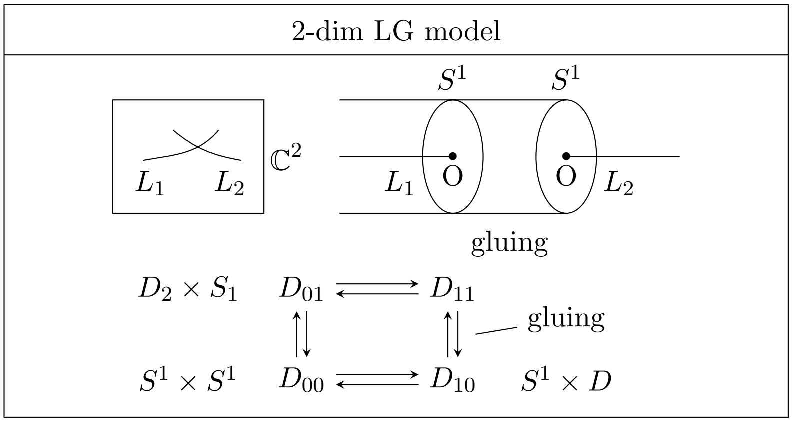
1. 1-dim LG model

1-dim LG model

$$\mathrm{D^b}(\mathbb{P}^2) \longrightarrow FS(\mathbb{C}^{*2}, x + y + \tfrac{1}{xy})$$

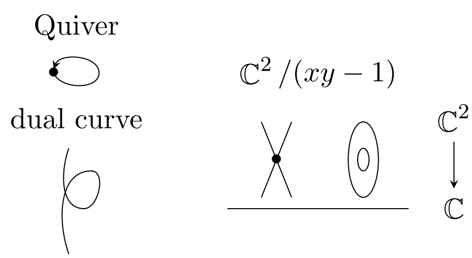


2. 2-dim LG model

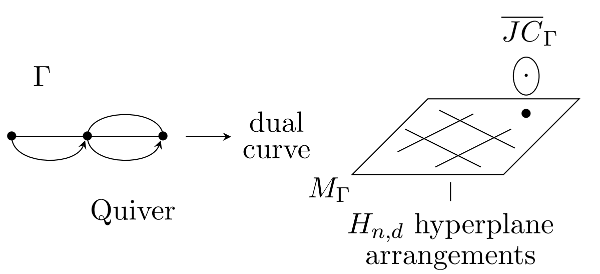


Example 3.4.

We give one more example of applications of sheaves of categories - the example of stability Hodge structures.



The general picture is:



The above M_Γ can be used as a LG mirror of the category of representations of the quiver Γ.

Conjecture 3.5. $\mathbb{C}^k/\cup H_{d,n}$ is the moduli space of stability conditions.

Definition 3.6. We call

$$\begin{array}{rl}\mathcal{H}: & \overline{JC}_{\Gamma} \longrightarrow \mathbb{C}^k \,/ \cup H_{d,n} \\ & \downarrow \\ & \mathbb{C}^n\end{array}$$

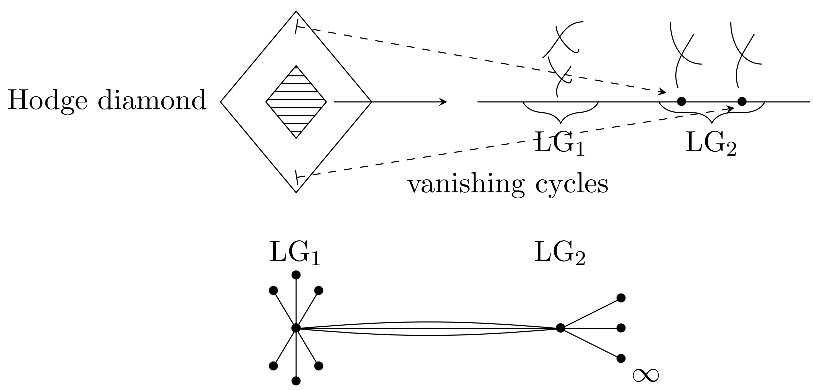
a **stability Hodge structure**.

Theorem 3.7 (Torelli)**.** *$\mathcal{H}$ recovers category Γ.*

Proof. It follows from the definition of sheaves of categories in multi-dimensional LG models. $\square$

Corollary 3.8. *$\mathcal{H} : \overline{JC}_{\Gamma} \longrightarrow \mathbb{C}^k \,/ \cup H_{d,n}$ recovers the Orlov spectrum of Γ.*

For A_n and $A_n / < \Gamma_1, \ldots, \Gamma_p >$, we checked this conjecture in [18], where a detailed account of Orlov spectra and its gaps are given.

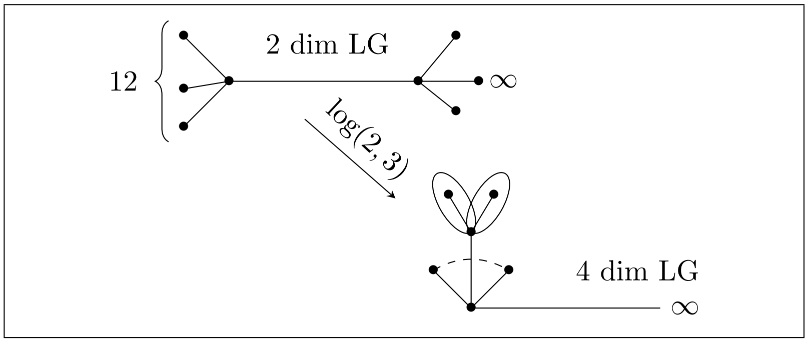


The theory of sheaves of categories suggests the following:

Definition 3.9. Let LG$_1$ be a part of a LG model which contains all vanishing cycles corresponding to all cohomologies but $h^{0,0}$, $h^{n,n}$. We call LG$_1$#LG$_2$ the **min topological sum of LG models**.

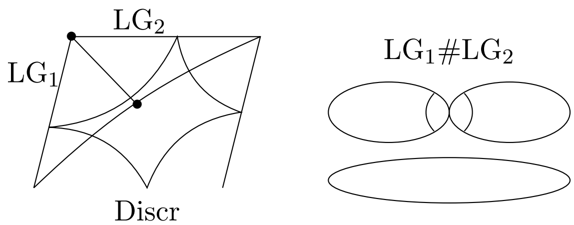
One of the most celebrated example of a LG model (non)stretching is the example of Dolgachev surface worked out by Donaldson in [7]. We interpret Donaldson's result from the point of view of sheaf of categories. We start with the change of the sheaf of categories structure on rational elliptic surfaces to Dolgachev surface - see below.

Table 4: Log transform

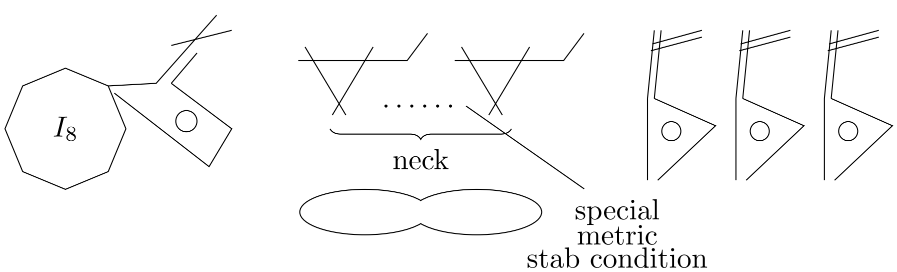


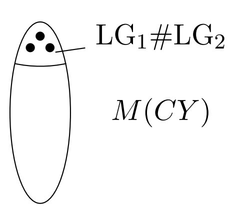
We give some more examples of the splitting of LG models. We start with the example of Hirzebruch surface F^1.

$$f : \mathbb{P}^1 \longrightarrow \widetilde{M_{st}}(CY)$$

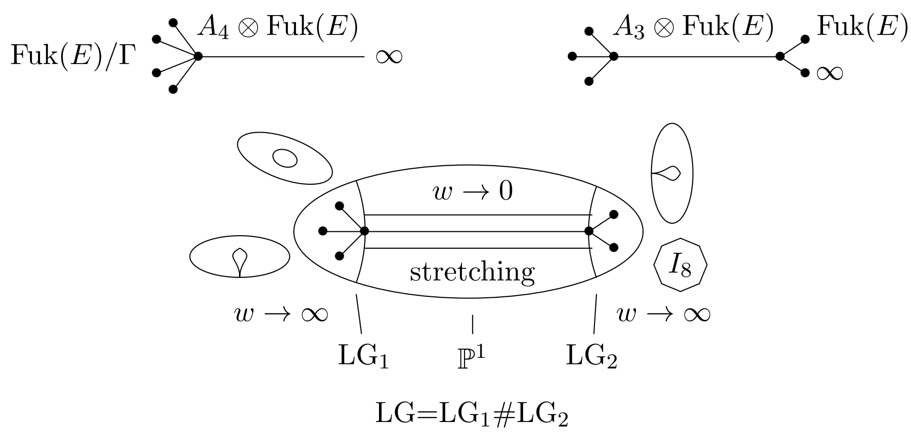


Example 3.10 (LG(F^1)).





Example 3.11 (LG of $F^1 = \mathbb{C}\,\mathbb{P}^2 \# \overline{\mathbb{C}\,\mathbb{P}^2}$)**.**



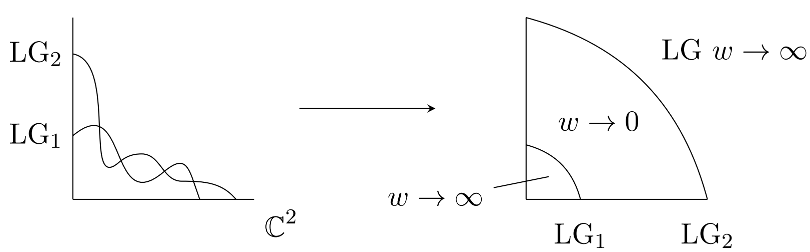
An important observation is the following conjecture.

Conjecture 3.12. The sheaf of categories associate with Dolgachev surface is not connected with a potential.

This is an analogue of Donaldson's statement that Dolgachev surface is not a connected sum of two 4-dimensional manifolds.

Similarly we can introduce the stretching in a 2-dimensional LG model too. The procedure of stretching the neck is the resolving of the singularities of the curves in the base of the 2-dimensional LG model. Creating obstructions to stretching as in the classical Seiberg-Witten theory can be done with surgeries - changing the Alexander polynomials - see [14]. We briefly describe the procedure below.

Example 3.13 (2-dim LG model)**.**



Observe that being a "stretched neck" is more than a semi-orthogonal decomposition. So to measure this we take Donaldson's point of view - use moduli spaces of objects. We suggest the parallel in conjecture 1.3.

A proof of this statement will lead to building an analogy between Donaldson theory and categorical Kähler metric. We summarize the analogy in the table below:

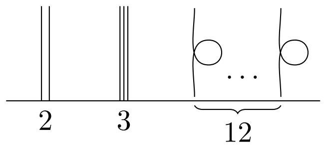
Table 5

<table>
<tr><td colspan="2">Categorical Donaldson theory</td></tr>
<tr><td colspan="2">Donaldson Theory
$\mathrm{H}^2(\mathcal{B}/\text{ASD connections}) \xrightarrow{\mu} \mathrm{H}^2(X)$
$\mathcal{B}$ - moduli space of ASD connections
M - compact moduli space
Donaldson Γ-invariant
$\Gamma() = 2\mu(M) + c_1(K_X)$</td></tr>
<tr><td colspan="2">Categorical theory</td></tr>
<tr><td>Classical</td><td>Categorical</td></tr>
<tr><td>$\mathcal{B}$
$\mathrm{H}^2(\mathcal{B}) \longrightarrow \mathrm{H}^2(X)$
$\cup$
Chamber structure

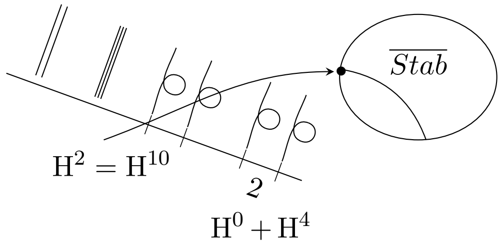
$\Gamma = \mu(M) + c_1$</td><td>$Met = \{\mathcal{E}, h\}$
$\overline{Met} \longrightarrow \overline{\mathrm{Stab}}(\mathcal{C})$
reducible metrizable object
$\downarrow$
limiting HN filtrations
$\cup$
Chamber structure
$\Gamma = \mu(M) + c_1$</td></tr>
</table>

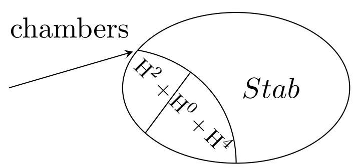
We also build the parallel with Bridgeland's theory of stability conditions.

Table 6: Example *Dolg* $2,3$

<table>
<tr><td colspan="2">


$Y = \widehat{\mathbb{P}_2}^{p_1,\dots,p_9}$
$\Gamma() = c_1(K_Y)$
Z - *Dolg* $2,3$
$\Gamma() = c_1(K_Z) + 7c_1(K_Z)$



</td></tr>
<tr><td>Classical
Donaldson theory
Chamber structures and
reduced connections</td><td>


reduced metrizable
object= phantom</td></tr>
<tr><td>Γ-invariant
non-rationality
for surfaces</td><td>Γ-invariant
non-rationality
in general</td></tr>
</table>

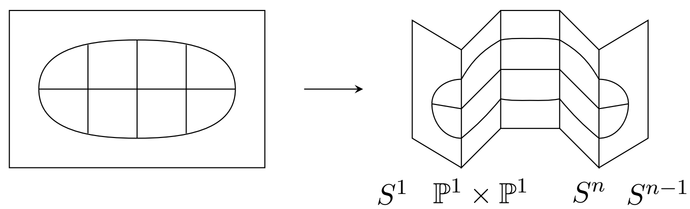
The approach with sheaves of categories can be useful in many directions. We give two applications connected with HMS. We start with the following categorification of classical Enrique-Petri theorem, see [11].

Let C be a trigonal canonical curve, $g(C) = g$. In $\mathbb{P}^{g-1}$, we have

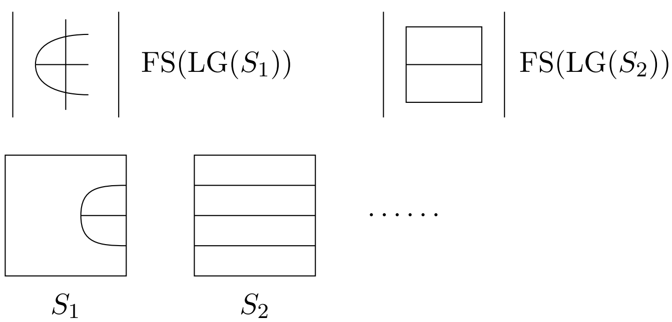
$$C \subset S = \bigcap_{c \subset Q_i}, \tag{10}$$

here Q_i are quadrics in $\mathbb{P}^{g-1}$ and S is the ruled surface. We describe a procedure of building HMS for genus g curve which is mod a complete intersection.

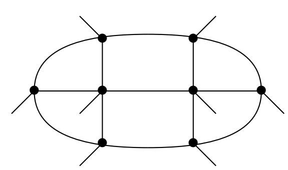
Step 1. We degenerate S to the union of $\mathbb{P}^1 \times \mathbb{P}^1$.



Step 2. Now we apply [1] to each S^1. We get sheaves of categories.



Step 3. Then we regenerate gluing sheaves of categories and using [1].



We get the HMS for C. Similarly we can do it for any canonical curve using Enrique-Petri theorem.

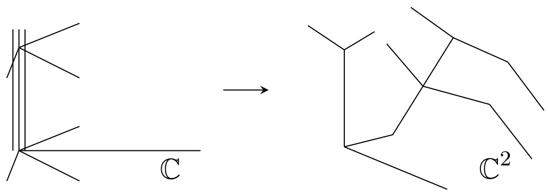
Theorem 3.14. *The procedure above proves HMS for canonical curves.*

The procedure above can be seen as a part of a much more general procedure of arborealization introduced by Nadler [22].

Let X be a 3-dim Fano and $\mathrm{LG}_1(W \to \mathbb{C})$ is 1-potential Landau-Ginzburg model and $\mathrm{LG}_2(W \to \mathbb{C}^2)$ is Landau-Ginzburg with two potentials.

Theorem 3.15. *The transform from sheaf of categories $LG_1(W \to \mathbb{C})$ to $LG_2(W \to \mathbb{C}^2)$ is a sheaf of categories version of arborealization.*

For more see [22] and [14].



Arborealization

4 Categorical Kähler metrics

In this section we briefly describe the categorical Kähler metric. The notion was developed in [15]. We recall the definitions and explain Table 1 from the introduction.

The idea of Kähler metric of a category imitates the classical definition up to a point.

Let C be a triangulated category. We define a new category C^0. Object of $C^0 := \mathcal{E}, h : \xrightarrow{\otimes_{\mathcal{O}_K} K} \mathcal{E}$, where h is a metric on $\mathcal{E}$.

$$Met(\mathcal{E}, h) = \left\{ \widetilde{\mathcal{E}} \in \operatorname{Ob} \mathcal{C}^0 + iso\ h : \widetilde{\mathcal{E}} \otimes_{\mathcal{O}_K} K \sim \mathcal{E} \right\} \tag{11}$$

We give a definition of categorical Kähler metric - it is a bit of cloud which allows us to define moduli spaces of objects.

Definition 4.1. Let C be a triangulated category and let $Z\colon K(C) \to \mathbb{C}$ be a central charge. We define the moduli space $Met(\mathcal{E}, h)$ as above. We say that C is a **categorical Kähler metric** if C is enhanced with the following data:

$D(1)$ Function $Mass : Met(\mathcal{E}, h) \to \mathbb{R}$.

$D(2)$ Flow $\mathcal{F}\colon Met(\mathcal{E}, h) \to \mathbb{R}$.

$D(3)$ Two other functions $Amp_- \leq Amp_+ : Met(\mathcal{E}, h) \to \mathbb{R}$.

$D(4)$ Potential function $S : \operatorname{Ob}(\mathcal{C}^0) \to \mathbb{C}$.

(additional care is needed to make data from different t-structures compatible and make $Met(\mathcal{E}, h)$ a nice space) satisfying the following axioms:

$A(1)$ The triangulated structure on C is recorded by the following actions: $\mathbb{Z}$-shift, $\mathbb{R}_{\geqslant 0}$-flow, $\mathbb{R}$-rescaling. They form a group $\mathbb{Z} \ltimes (\mathbb{R}_{\geqslant 0} \times \mathbb{R})$.

$A(2)$ These actions are compatible with the functions Mass, Amp_-, Amp_+, S.

$A(3)$ Additivity of all these functions on $Met(\mathcal{E}, h)$, namely:

$Z((\mathcal{E}, h), (\mathcal{F}, h)) = Z + Z$;

$S^{\mathbb{C}}(\oplus) = S^{\mathbb{C}} + S^{\mathbb{C}}$;

$\mathrm{Amp}_-(\oplus) = \min(\mathrm{Amp}_-,\mathrm{Amp}_-)$;

$\mathrm{Amp}_+(\oplus) = \min(\mathrm{Amp}_+,\mathrm{Amp}_+)$.

$A(4)$ Qualitative properties - fixed points of the flow.

$\mathrm{Mass}(\mathcal{E}, h) \geqslant Z(\mathcal{E})$

Mass$= |Z(\mathcal{E})| \leftrightarrow \exists!\ \theta \in \mathbb{R}$, F - $\theta\mathbb{R} = 0$ on $(\mathcal{E}, h)$. Then $\mathcal{E}$ is stable.

$\forall(\mathcal{E}, h)$, $\lim\limits_{t\to\infty} e^{\mathcal{F}t}(\mathcal{E}, h)$, $\exists$ in the flag compactification

$$(\mathcal{E}_1, h_1), \ldots, (\mathcal{E}_n, h_n).$$

$\forall i, \exists!\ \theta_i,\ \theta_1 > \cdots > \theta_n$, $(\mathcal{F} - \theta_i\mathbb{R})|_{(\mathcal{E}_i, h_i)} = 0$. We call these fixed points **Harder–Narasimhan filtrations**.

$A(5)$ There exists a compactification of the moduli space of Z-stable objects which consists of HN filtrations.

These axioms were used in [15] to prove the existence of compact moduli spaces of stable objects. We will use these moduli spaces to build categorical Donaldson invariants.

5 Categorical base loci and categorical Okounkov bodies

In this section we introduce the notions of categorical linear systems and categorical base loci. We connect these notions with gaps of spectra of categories.

From what we have said it becomes clear that Okounkov bodies play an important role in studying the complexity of functors. In order to make it suitable to our noncommutative birational approach we devise a categorical analogue of the notion of Okounkov body $\Delta(D)$. Indeed, the notions of divisors and their sections and multiplicities can be immediately translated into categorical language. The flag of submanifolds becomes a flag of subcategories. The sections become natural transformations between spherical and restriction functors, and valuations ν_i correspond to how far along one can lift these natural transformations - see the Figure 2 and Figure 3 below. In these figures S is the restriction functor and t is a spherical functor of a twist by a divisor.

$$\begin{tikzcd}
\cdots \arrow[r, "S"] & \{-3R_1\} \arrow[r, "S"] & \{-2R_1\} \arrow[r, "S"] & \{-R_1\} \arrow[r, "S"] & Id \\
 & & & & \{-D\} \arrow[u, "t"] \arrow[ull] \arrow[ul, dotted]
\end{tikzcd}$$

Figure 2: Step 1 ($\nu_1 = k_1 = 2$)

Definition 5.1. Following the figure above we define ν_1 as the maximal number of liftings of the natural transformation t.

$$\begin{tikzcd}
\cdots \arrow[r, "S"] & \{-3R_2\} \arrow[r, "S"] & \{-2R_2\} \arrow[r, "S"] & \{-R_2\} \arrow[r, "S"] & Id_{R_1} \\
 & & & & \{-k_1R_1 - D\} \arrow[u, "t"] \arrow[ulll] \arrow[ull, dotted] \arrow[ul, dotted]
\end{tikzcd}$$

Figure 3: Step 2 ($\nu_2 = k_2 = 3$)

Similarly we define ν_2 as the maximal number of liftings in the figure above. In the same way we define ν_i using the flag of subvarieties $R_1, ..., R_d$ or of subcategories.

Remark 5.2. Definition 5.1 is a categorification of the usual definition of Okounkov body. Classically k_i is the multiplicity with which D passes through R_i.

We give several examples of flags of categories mainly coming from derived categories of flags of subvarieties. The cube of categories below is given by two quadrics Q_1 and Q_2 in $\mathbb{P}^3$ and their intersection - an elliptic curve E. As shown in Figure 4, derived categories of Q_i and E define a flag of categories R_1, R_2.

$$\begin{tikzcd}
 & \mathrm{D^b}(X) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & \mathrm{D^b}(Q_2) \arrow[dl] \arrow[dd] \\
\mathrm{D^b}(Q_1) \arrow[rr] \arrow[dd] & & \mathrm{D^b}(Q_1 \cap Q_2) \arrow[dd] & \\
 & \mathrm{D^b}(Q_3) \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{D^b}(Q_2 \cap Q_3)) \arrow[dl] \\
\mathrm{D^b}(Q_1 \cap Q_3) \arrow[rr] & & \mathrm{D^b}(Q_1 \cap Q_2 \cap Q_3) &
\end{tikzcd}$$

Figure 4: B side

The Homological Mirror Symmetry defines equivalent Fukaya-Seidel (FS) categories (see e.g. [2], [20]) with a mirror cube given below in Figure 5 as well as the mirror of the flag of derived categories.

$$\begin{tikzcd}
 & \mathrm{FS}(Z, w) \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & \mathrm{FS}(Z_2, w_2) \arrow[dl] \arrow[dd] \\
\mathrm{FS}(Z_1, w_1) \arrow[rr] \arrow[dd] & & \mathrm{FS}(Z_{12}, w_{12}) \arrow[dd] & \\
 & \mathrm{FS}(Z_3, w_3) \arrow[rr, dotted] \arrow[dl, dotted] & & \mathrm{FS}(Z_{23}, w_{23}) \arrow[dl] \\
\mathrm{FS}(Z_{13}, w_{13}) \arrow[rr] & & \mathrm{Fuk}(E) &
\end{tikzcd}$$

Figure 5: A side

Definition 5.3. Consider a flag of categories $R_1, ..., R_d$. For $m \in \mathbb{Z}_{>0}$, denote $t^{\circ m}$ by $m \cdot t$. Now we can define the $\mathrm{Im}(m \cdot t)$ as $(\nu_1, ..., \nu_d)$, where every ν_i depends on $m \cdot t$.

As a result we have:

Definition 5.4. We define $\delta(t)$ as the closed convex hull of

$$\lim_{m \to \infty} \frac{1}{m} Im(m \cdot t).$$

$\delta(t)$ is a categorical notion. We start with two functors. $\delta(t)$ measures how these two functors interact asymptotically with respect to a flag of subcategories. We will give more examples later. The categorical Okounkov body will play an important role in classifying the base loci of a category.

Remark 5.5. Observe that we get some modifications of Okounkov body if we additionally twist divisor D by a multiplier ideal sheaf - test configuration. In fact we get a sequence of Okounkov bodies associated with the filtration on the sheaf of ideals. For more see [18].

6 Categorical multiplier ideal sheaves and categorical Kähler-Einstein metrics

In this section we present a scheme for building theory of categorical Kähler-Einstein metrics. We start with examples of categorical multiplier ideal sheaves.

Categorical Multiplier Ideal Sheaf for A_n:

One of the main observations of [18] is that the spectra of A_n can be interpreted from the point of view of categorical multiplier ideal sheaves for the category A_n. We have a sheaf of generators (a sheaf of localized categories) for which the jump numbers determine how many sides do we take from the whole polygon in order to form the forbidden part.

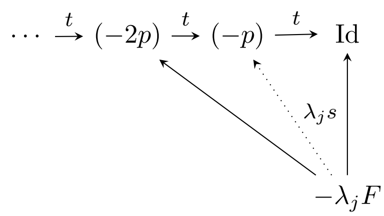
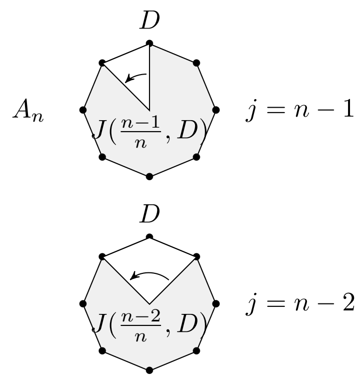
We record our observation in the following theorem - see also Table 8.

Theorem 6.1. *The multiplier ideal sheaf for the category A_n and the localization functor - restricting an n-gon has jump numbers $\frac{n-1}{n}, \ldots, \frac{1}{n}$. The multiplier ideal sheaf $J(\lambda_1, \ldots, \lambda_k)$ determines the Orlov spectrum of A_n.*

The proof of this theorem is a direct consequence of the definition of $J(\lambda_1, \ldots, \lambda_k)$ - see Table 8.

In this case the categorical multiplier ideal sheaf is a sequence of localizations $J(C_k, \lambda_k F) \subset \cdots \subset J(C_1, \lambda_1 F)$. Marking a polygon corresponds to localizing by a subcategory. The localization by the biggest polygon produces the first non-trivial category $J(C_k, \lambda_k F)$ and by the smallest $J(C_1, \lambda_1 F)$. In the table below we represent the multiplier ideal sheaf in this case as rotation by angles of λ_j of the localization functor F.

Table 7

The Mult. Ideal Sheaf	Functor Localization by A_n
	

As the proof of Theorem 6.1 shows obtaining Orlov spectra is moving up on marking polygons - see the table below. This is the baby K-calculus.

Table 8

Spectra	Sheaves and Jump numbers
$\{\lfloor\frac{n-1}{2}\rfloor\} \cup \{n-1\}$	$\lambda_n = \frac{n-1}{n}$
$\{(\lfloor\frac{k-1}{2}\rfloor),\ ...,\ \lfloor\frac{n-1}{2}\rfloor\}$ $\cup\{k-2,\ ...,\ n-1\}$	$\lambda_k = \frac{k}{n}$
$\{0,\ 1,\ \ldots,\ n-1\}$	$\lambda_1 = \frac{1}{n}$

As we have seen the Okounkov body and the multiplier ideal sheaf can be made totally categorical. (Showing that categorical valuations we have defined satisfy the usual equalities requires additional work - see [14].) Classically these two notions have been used to define:

1. Futaki invariants - integral of functions (defining testing configurations) over Okounkov bodies for a line bundle L. If these integrals over all these testing configurations are all positive we conclude that the smooth projective variety X has a Kähler–Einstein met-

ric.

2. The log canonical threshold - the smallest jump number. If a Fano manifold X of dimension n has a log canonical threshold bigger than $n/(n+1)$ then we conclude that smooth projective variety X has a Kähler–Einstein metric (see [6], [23]).

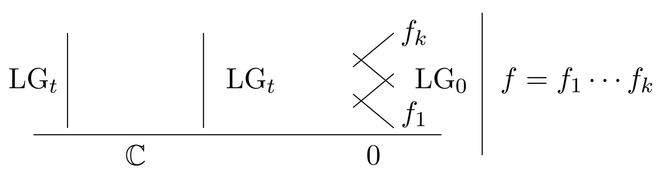
The categorical interpretation of the Futaki invariant is an integral over the categorical Okounkov body defined by a functor F (the categorical version of a twist by L) and a testing configuration - an additional twist by an ideal choosing faces of the categorical Okounkov body. We will call these special functors Landau-Ginzburg testing functors.

We introduce the definition:

Definition 6.2. We will call a category a **Kähler–Einstein category** if all categorical Futaki invariants for all Landau-Ginzburg testing functors are positive.

Recall that:

Definition 6.3 (Testing LG functors)**.** We call a family of LG functors a **testing configuration**

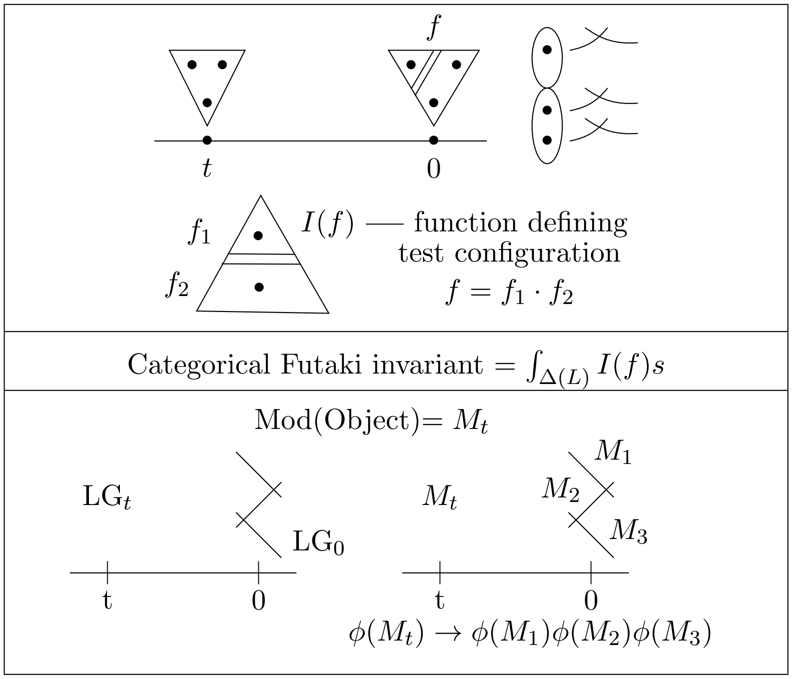


compatible with $\mathbb{C}^*$-action iff

1. $\forall t \neq 0$, LG_t are isomorphic;

2. LG_0 consists of several LG models.

Example 6.4 ($\mathrm{LG}(\mathbb{P}^2)$)**.**



We collect the Kähler–Einstein correspondences in Table 9 below.

Table 9

Classical Kähler-Einstein	Categorical Kähler-Einstein
X Fano, $\dim(X) = n$ $\exists$ Kähler-Einstein metric	Category of Kähler-Einstein type
$\forall\ f_\lambda$, testing family $\int_{\Delta(L)} f_\lambda > 0$	$\int_{\Delta(L)} I(F) > 0$ I - testing LG functor
$\lambda_1 > \dfrac{\dim X}{\dim X + 1}$	$\lambda_1 > \dfrac{\dim X}{\dim X + 1}$? or Orlov spectra?

Let X, K_X be a maximal degeneration. We formulate the following conjectures:

Table 10: Correspondences

Classical	Categorical
$X, K_X \to X_0$ maximal degeneration	$D_n \subset \cdots \subset D_1 \subset D$ sequence of Localizations of LG models
HN Filtrations for X_0	Categorical HN Filtrations for categories
$X, K_X \to X_0, \mathrm{Ext}$	Orlov Spectra of $\mathrm{Fuk}(X)$

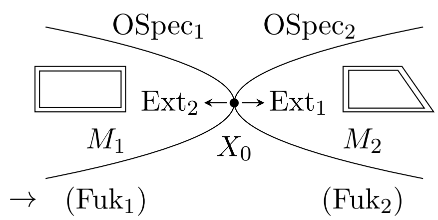
We have:

Conjecture 6.5. Degenerations $X, K_X \to X$ correspond to HN filtrations for categorical KE metrics.

Conjecture 6.6. There exists a canonical degeneration $X, K_X \to X$ among HN filtrations for categorical KE metrics.

Conjecture 6.7. There exists a categorical invariant Φ associated with canonical degeneration $X, K_X \to X$.

Example 6.8 (Horikawa's)**.**



The categorical invariants Φ will be defined in the next section.

7 Applications

In this section we briefly introduce the moduli space of stable objects of Fukaya-Seidel categories of Landau-Ginzburg models (and their degenerations). We also introduce a categorical version of Donaldson's invariants. At the end we discuss some applications. We refer to [8] for classical definitions.

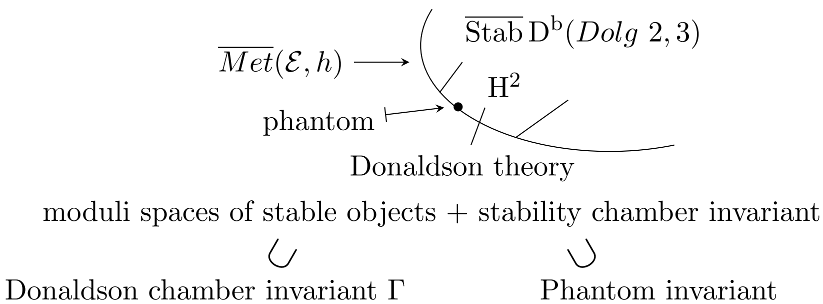
7.1 Geometric applications

Assume that $\mathcal{C}$ is a triangulated category with stability conditions, e.g. $\mathcal{C}$ is $\mathrm{D^b}(X)$, $\mathrm{Fuk}(X)$, $\mathrm{FS}(X)$.

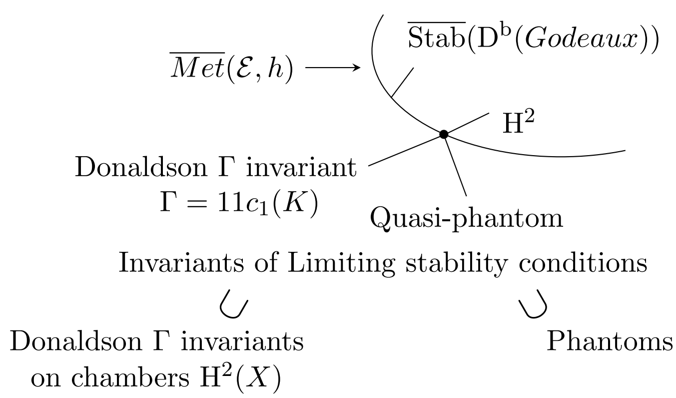
Conjecture 7.1. Donaldson theory of the chamber invariant Γ has an analogue for compactified moduli spaces of stable objects for $\overline{\mathrm{Stab}}(\mathcal{C})$. We correspond conjecturally the chamber structure on $\mathrm{H}^2(X)$ to the chamber structure of $\overline{\mathrm{Stab}}(\mathcal{C})$.

Here we use the wall structure for moduli spaces of stability conditions. We look at some examples.

Example 7.2 (Dolgachev Surface)**.**

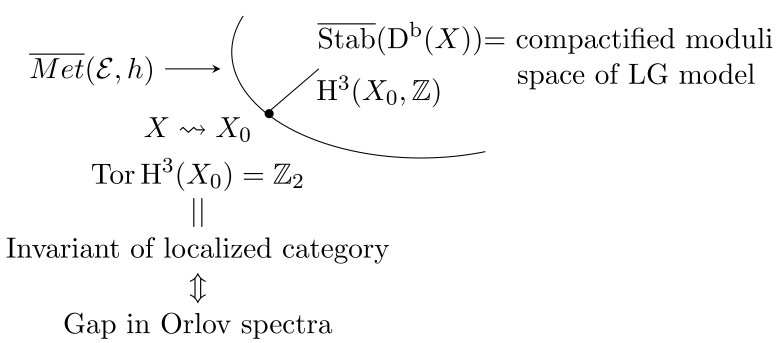


Example 7.3 (Godeaux surface)**.**



3-dimensional examples:

Example 7.4 (Artin-Mumford example)**.**



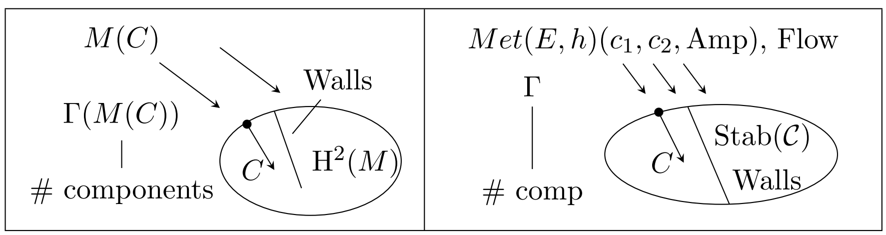
We first build a parallel between:

1. Classical Donaldson theory and theory of categorical Kähler metrics.

2. Classical collapse for KE metrics and HN filtrations for KE metrics for FS category.

Categorical metrics and categorical Donaldson theory

The theory of moduli spaces of objects is a consequence from the categorical metrics. The moduli space $Met(E,h)(c_1,c_2,\mathrm{Amp})$ is the fixed point set of the Flow. We explore this moduli space and define categorical Donaldson invariants. For more see [15].

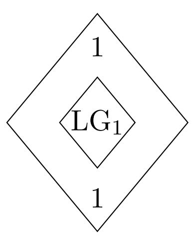


Conjecture 7.5.

$$\begin{array}{ccc} \{Met(Comp) \neq 0\} & \cong & \{\text{Voisin non-splitting on diagonal}\} \\ \wr\| & & \|\wr \\ \{\text{Impossible to stretch the neck}\} & \cong & \{\text{Gap in Orlov spectra}\} \end{array}$$

We elaborate this table. Stretching the neck for Landau-Ginzburg models is the novelty we propose in this paper. This is a phenomenon stronger than Hodge theory. Stretching the neck and its obstructions serve as a way of putting together categories.

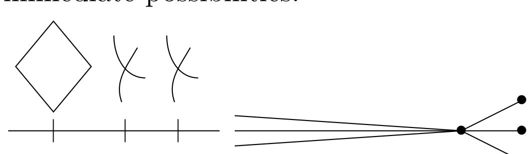
Conjecture 7.6. If X is rational, then $\phi(\mathrm{LG}_1\#\mathrm{LG}_2) = \phi(\mathrm{LG}_1)$.



dim=2 If $\phi(\mathrm{LG}_1\#\mathrm{LG}_2) \neq 0$, then X is not rational

dim=3
dim=4 $\phi(\mathrm{LG}_1\#\mathrm{LG}_2) \neq \phi(\mathrm{LG}_1)$

This gives immediate possibilities.

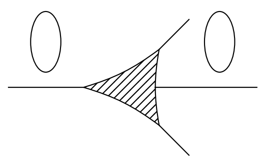


$\phi(\mathrm{LG}_1\#\mathrm{LG}_2) \neq \phi(\mathrm{LG}_1)$

1. Obstruction to splitting of the diagonal.
2. Gaps in the dynamical spectra.
3. Gaps in Orlov spectra.

We restrict ourselves to the case when X is a smooth projective variety such that $h^{p,q}(X) \neq 0$ iff $p = q$. In this case extensions of Lagrangians L_1 , L_2 coming from LG_1 and LG_2 produce nontrivial moduli spaces of Lagrangians $\mathrm{Mod}(L_1, L_2)$.

$$1 \longrightarrow L_1 \longrightarrow \mathcal{E} \longrightarrow \varphi(L_2) \longrightarrow 1 \tag{12}$$



$\mathrm{HF}(L_1, \varphi(L_2)) = \mathbb{C}$

More generally,

$$\dim \mathrm{Mod}(L_2) < \dim \mathrm{Mod}(L_1, L_2). \tag{13}$$

In the case of 3-dim Fano $\mathrm{D^b}(Y) =< B_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) >$ we have an acyclic instanton

$$1 \to E \to \widetilde{E} \to \mathcal{O}_Y^{n-2} \to 1, \tag{14}$$

where $E \in (\mathcal{O}_Y(1))^{\perp}$. Here $L_1 = E$, $L_2 = \mathcal{O}_Y^{n-2}$.

Conjecture 7.7. Assume that $B_Y \neq \mathrm{D^b}(C)$, then $Don(\mathrm{D^b}(Y)) = Don(\ )\sinh(t)$. So it is a basic class which does not correspond to a blow-up.

Conjecture 7.8.

1. $\mathrm{HF}(L_1, \varphi(L_2)) \neq 0$ is an obstruction to rationality.
2. $\dim \mathrm{Mod}(L_2) < \dim \mathrm{Mod}(L_1, L_2)$ is an obstruction to rationality.

This considerations suggest the following:

Conjecture 7.9. Let X be a smooth projective variety s.t. $h^{p,q}(X) \neq 0$ iff $p = q$. If $\mathrm{Mod}(L_1, L_2)$ is not trivial then X is not rational.

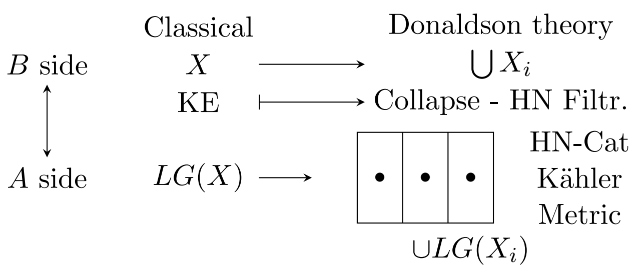
Idea of proof. Let us consider the case when X is 3-dimensional. Let X be rational. Then X is a blow-up of a projective space in a curve. The space $\mathrm{Mod}(L_1, L_2)$ is the mirror of the space of instantons - extensions of the ideal sheaf of the exceptional curve. Such an instanton is non-stable if rigid.

□

We have the opposite conjecture too. It comes from the theory of KE categories and the HN filtrations.

Conjecture 7.10. Let X be a smooth projective variety such that $\mathrm{D^b}(X) =< E_1, \ldots, E_n >$. Then $X \overset{\mathrm{bir}}{=} \mathbb{P}^N$.

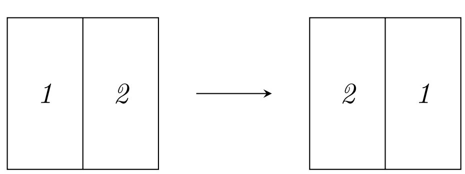
Idea of Proof.



□

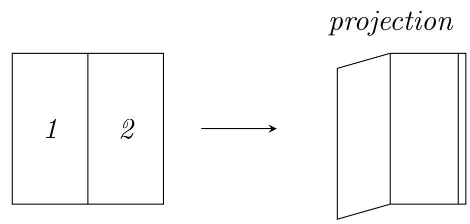
Proposition 7.11. *We have the following two operations:*

1.



Mutation - Change of Stability

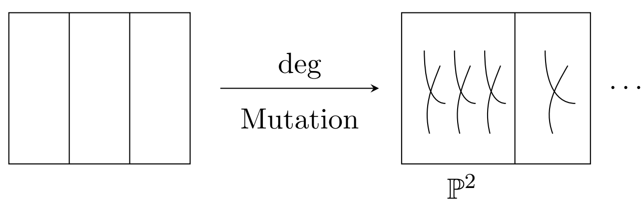
2.



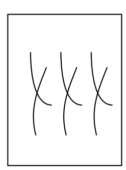
Change of HN $\to$ *Change of Categories*

Idea.

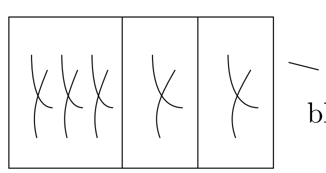
dim 2 NH for Categorical Kähler Metric.



Step 1 Via further degeneration

Cat  $\longrightarrow$ As it follows from [12] this is $\mathbb{P}^2$.

Step 2

 adding further blocks - blow-ups

Step 3 Regeneration HN $\to$ Categorical Kähler Metric.

$\square$

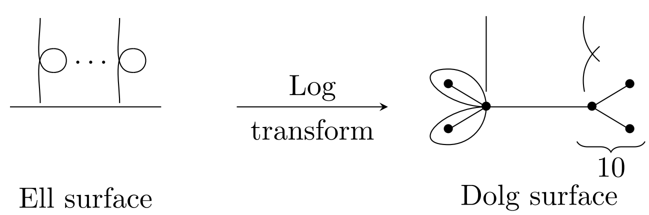
The above observations lead to a general program. Let us concentrate on the B side. The stability condition is given by

(15)
$$\int_X \operatorname{tr}(e^{\frac{iw}{\hbar}+F})\Gamma(T_X) \xrightarrow{\hbar\to 0}$$
$$\left(\frac{1}{n!}\left(\frac{1}{\hbar}\right)^{\dim X} w^n \operatorname{rk}\mathcal{E} + \frac{1}{n-1}\left(\frac{1}{\hbar}\right)^{\dim X-1} w^{n-1}c_1 + \cdots\right)\Gamma$$

We have the following correspondence between classical Donaldson theory and categorical Donaldson theory:

$$\begin{aligned} &Mod(Stab) \xrightarrow{\hbar\to 0} Mod(c_1,c_2)\\ &Don(Stab) \xrightarrow{\hbar\to 0} Don(c_1,c_2) \end{aligned} \tag{16}$$

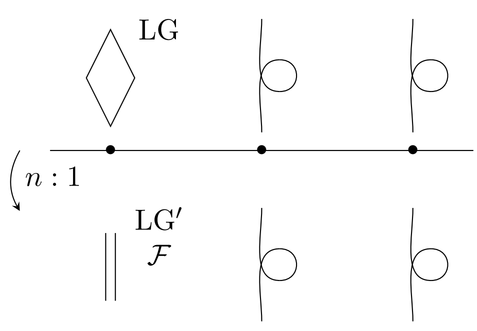
Recall that:



$$\Delta_K^{e^{2[T]}} Don(Ell) = Don(Dolg\ 2,3), \tag{17}$$

where Δ_K is an Alexander polynomial and T is a torus fiber.

We have similar examples in dimension 3.



$$Don(LG) \longrightarrow Don(LG')\cdot \Delta_{\mathcal{F}}(\mathrm{D^b}(K3)). \tag{18}$$

Here $K3$ is the fiber of the LG model and $\Delta_{\mathcal{F}}$ is the Alexander polynomial of the functor $\mathcal{F}$.

Here we apply several incarnations of conjecture 1.3 in the introduction.

Conjecture 7.12.

$$Don(\widehat{\mathbb{P}^3_C}) = Don(\mathbb{P}^3)\cdot Don(Rul(C)), \tag{19}$$

where $Rul(C)$ is the ruled surface over C.

Conjecture 7.13. In dimension 4,

$$Don(\widehat{\mathbb{P}^4_S}) = Don(\mathbb{P}^4)\cdot Don(Rul(S)). \tag{20}$$

Classical	Categorical
$\begin{array}{c} u \\ \downarrow \\ \mathrm{mod}\ \times X \end{array}$ $\begin{array}{c} \mathrm{Ch}(u) \\ \uparrow \\ \mathrm{H(Mod)}\ \times\ \mathrm{H}^2(X) \end{array}$	$\begin{array}{c} u \\ \downarrow \\ \mathrm{mod}\ \times \mathrm{D^b}(X) \end{array}$ $\begin{array}{c} \mathrm{Ch}(u) \\ \downarrow \\ \mathrm{H(Mod)}\ \times\ \mathrm{HP}(\mathrm{D^b}(X)) \end{array}$

We have $\mathrm{D^b}(X) =< \mathcal{A}, E_1, \ldots, E_n >$.

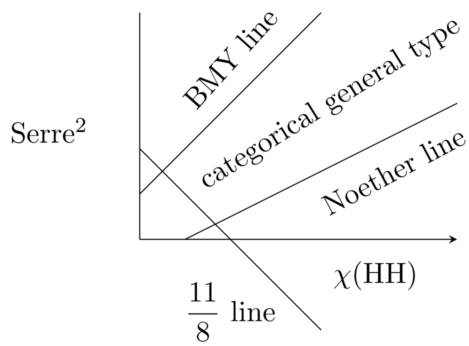
Conjecture 7.14.

$$Don(\mathrm{D^b}(X)) = Don(\mathcal{A})\cdot Don(E_1, \ldots, E_n). \tag{21}$$

This allows us to connect birational geometry of 4-dimensional Fano with the theory of 2-dimensional categories.

Let X be a 4-dimensional Fano e.g. 4-dimensional cubic. The formula is:

$$\begin{aligned} \int_X \mathrm{tr}\, e^{\frac{iw}{\hbar}+F}\Gamma = & \frac{1}{n!}\Big(w^{\dim X}\,\mathrm{rk}(E)\Big) + \\ & \frac{1}{(n-1)!}\Big(w^{\dim X+1}c_1(E)\Big) + \\ & \frac{1}{(n-2)!}\Big(w^{\dim X+2}c_2(E)\Big) + \cdots . \end{aligned} \tag{22}$$



Conjecture 7.15.

$$Don(X^4_{\text{cubic}}) = Don(\mathcal{A})(\cosh\cdot\, l)^3. \tag{23}$$

Here

$$Don(\mathcal{A}) = \exp(Q_R)\prod_{i=1}^{3}\frac{\sinh(F_i)}{\sinh(F_i/p)\sinh(F_i/q)}. \tag{24}$$

The reason for this is

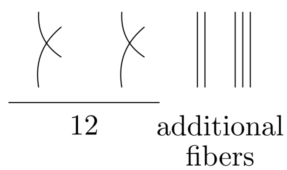
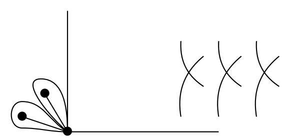
$$\mathrm{Fuk}(E\times E/\mathbb{Z}_3) = \mathrm{D}^{\mathrm{b}}(E(2)(1,2,1,2,1,2)). \tag{25}$$

So $p_1 = p_2 = p_3 = 1$, $q_1 = q_2 = q_3 = 2$.

Conjecture 7.16. $Don\,\mathrm{D}^{\mathrm{b}}(X^4_{\text{cubic}})$ cannot be related to $Don\,\mathrm{D}^{\mathrm{b}}(\widehat{\mathbb{P}^n_S})$, where S is a surface of general type.

The main idea behind this conjecture is the conjecture 1.3.

It will be interesting to define basic classes for LG models.

<table>
<tr><th>Classical</th><th>Categorical</th></tr>
<tr><td>Basic classes</td><td>Vanishing cycles</td></tr>
<tr><td>gluing surfaces via Donaldson invariants</td><td>gluing categories via categorical Donaldson invariants</td></tr>
<tr><td colspan="2">Examples</td></tr>
<tr><td>Basic classes F

</td><td>Vanishing cycles
</td></tr>
</table>

Question 7.17.

- Define categorical Bogomolov-Miyaoka-Yau inequalities.
- Noether inequalities.
- $\dfrac{11}{8}$ conjecture.

Gluing of Donaldson invariants is parallel to CH^0 (non)triviality in Voisin theory.

Voisin theory	Categorical theory
$X \xrightarrow{deg} X_0$ $\mathrm{Br}(X_0) \neq 0$ $\Rightarrow X$ is stably non-rational	$\mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X_0)$ Cat $\mathrm{Br}(X_0) \neq 0$ $\Rightarrow X$ is stably non-rational

Conjecture 7.18. Let $\mathcal{C}_b$, $b \in B$ be a flat family of categories s.t. $\exists\, U \subset B \ni 0$, $\mathcal{C}_b = \mathrm{D^b}(X_b)$ and Cat $\mathrm{D^b}(X_0) \neq 0$. Then there exists Zariski open $U \subset B$ s.t. for very general $b \in U$, X_b is stably non-rational.

Conjecture 7.19 (dim 4)**.** If $Don(\mathrm{D^b}(X_0))$ acquires three additional basic classes then there exists Zariski open $U \subset B$ s.t. for very general $b \in U$, X_b is stably non-rational.

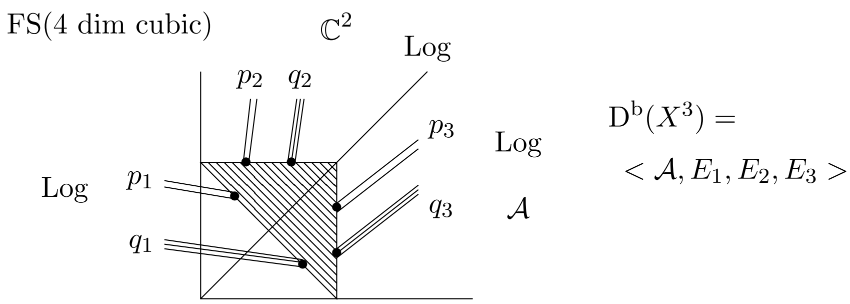
As we have mentioned, the theory of sheaves of categories is a convenient way of computing Donaldson invariants.

Classically, Gompf and Mrowka perform 3 pairs of Log transforms.

$$\overbrace{S^1 \times S^1 \times S^1}^{p_3, q_3} \times S^1$$
$$p_1, q_1 \quad p_2, q_2$$

(26) $$Don(E(2), p_1, q_1, p_2, q_2, p_3, q_3) = \exp(\frac{Q}{2}) \prod_{i=1}^{3} \frac{\sinh(F_i)^2}{\sinh(\frac{F_i}{p_i})\sinh(\frac{F_i}{q_i})}.$$

In the case of 4-dim cubic, we have a LG model.



(27) $$Don(\mathrm{D^b}(X^3)) = Don(\mathcal{A})e^{Q_i}$$

(28) $$Don(\mathcal{A}) = Don(E(2), p_1, q_1, p_2, q_2, p_3, q_3)$$

This is our most important application of sheaves of categories construction.

1. Log transform p_1, q_1
   - 1-basic class
   - 1-phantom
   - 1 additional generation time
2. Log transform
   - 2-basic class

   - 2-phantom
   - 2 additional generation times

3. Log transform
   - 3-basic class
   - 3-phantom
   - 3 additional generation times

$$\begin{tikzcd}
 & \mathrm{FS}_{X_3} \arrow[rr] \arrow[dd, dotted] & & \mathrm{FS}_{X_2} \arrow[dd] \\
\mathrm{FS}_{X_4}(\mathbb{C}^4, w) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{FS}_{X_3}(\mathbb{C}^3, w) \arrow[ur] \arrow[dd] & \\
 & \mathrm{FS}_{X_2} \arrow[rr, dotted] & & \mathrm{Fuk}(E) \\
\mathrm{FS}_{X_3}(\mathbb{C}^3, w) \arrow[ur, dotted] \arrow[rr] & & \mathrm{FS}_{X_2}(\mathbb{C}^2, w) \arrow[ur] &
\end{tikzcd}$$

The structure of sheaves of categories suggests the following procedure of computing Donaldson invariants.

Conjecture 7.20.

$$Don(\mathrm{FS}_{X_4}) = Don_{X_2}()\cdot \text{(function of } (t_1, t_2, t_3))\cdot Don(), \tag{29}$$

where t_1, t_2, t_3 are basis classes.

The above considerations in connection with [18] suggest the following correspondence.

$$\{\text{Basic classes}\} \longleftrightarrow \{\text{Gaps in Orlov spectra}\}$$

Conjecture 7.21. The creation of 3 basic classes leads to non-rationality of 4-dimensional cubic.

This conjecture suggests that the gap of the Orlov spectrum of $\mathrm{D}^{\mathrm{b}}(X) \geqslant 3$ if and only if X is non-rational.

Conjecture 7.22. We have the following conjecture:

$$
\begin{tikzcd}
\left\{\begin{array}{c}\text{Dependence}\\ \text{of}\\ \text{basic classes}\end{array}\right\} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \left\{\begin{array}{c}\text{Noether}\\ \text{Lefschetz}\\ \text{loci}\end{array}\right\} \arrow[dl, leftrightarrow] \\
& \left\{\begin{array}{c}\text{Divisors}\\ \text{of jumps of}\\ \text{the gaps}\end{array}\right\} &
\end{tikzcd}
$$

We develop further the connection with Voisin theory.
In the case $\dim X = 3$, we have Voisin theory:

$$X \xrightarrow{deg} X_0$$

$$
\begin{array}{c}
\operatorname{Tor} \mathrm{H}^3(X_0, \mathbb{Z}) \\
\cap \\
K = \wedge \mathrm{H}^{\mathrm{odd}} \oplus S\, \mathrm{H}^{\mathrm{even}} \\
A(K) \longrightarrow \mathbb{Z} \\
\updownarrow \\
\text{new basic class} \\
\parallel \\
\text{gap in Orlov spectra}
\end{array}
$$

So we formulate:

Conjecture 7.23. Let $\mathcal{C}_b$ be a flat family of categories, where $b \in B \ni 0$, so that $Don(\mathrm{D^b}(X_0))$ has an l additional basic classes for $l = dim(X_0) - 2$. Then $\exists\, U \subset B$ s.t. for very general $u \in U$ so that $\mathcal{C}_u = \mathrm{D^b}(X_u)$, X_u is stably non-rational.

We move to $\dim X = 4$. We start with $E(2)(p_1, q_1, p_2, q_2, p_3, q_3)$.

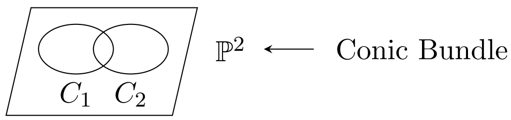
Theorem 7.24.

$$\mathrm{D^b}(E(2)(p_1, q_1, p_2, q_2, p_3, q_3)) \neq < \mathrm{D^b}(X_1), \mathrm{D^b}(X_2) >, \tag{30}$$

where X_1 and X_2 are algebraic surfaces.

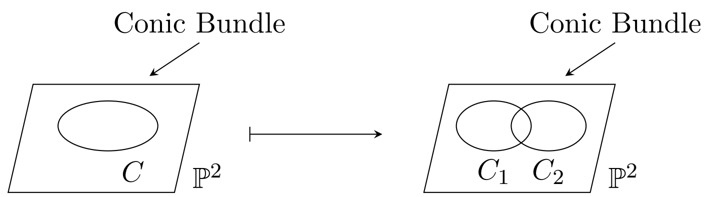
Proof. It follows from [13]. □

We move to special families - conic and quadric bundles.
We consider a three-dimensional conic bundle:



We consider a degeneration of conic bundles:

$$X \xrightarrow{\ \ deg\ \ } X_0.$$

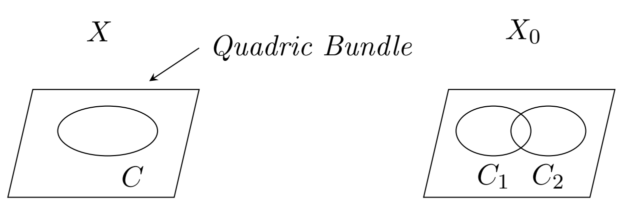


$$0 \neq \Gamma \in \mathrm{H}^1(C, \mathbb{Z}_2) \longrightarrow \begin{array}{l} \Gamma_1 \in \mathrm{H}^1(C_1, \mathbb{Z}_2) \\ \Gamma_2 \in \mathrm{H}^1(C_2, \mathbb{Z}_2) \end{array}$$

From the conjectures formulated above we get:

Theorem 7.25. *Assume that $Don(\mathrm{D}^{\mathrm{b}}(X_0))$ acquires two additional basic classes. Then this implies $\operatorname{Tor}\mathrm{H}^3(X_0, \mathbb{Z}) \neq 1$ and X_u is stably non-rational.*

Theorem 7.26. *Let X be a 4-dimensional quadric bundle.*



Assume that $Don(\mathrm{D}^{\mathrm{b}}(X_0))$ acquires three additional basic classes. Then X_u is stably non-rational.

Proof. Both theorems follow from [16]. □

Similarly to 2-dimensional Fukaya categories the same approach applies. Consider two Horikawa surfaces:

$$\mathrm{FS}(X_1/K_{X_1}) \xleftrightarrow[\text{surgery}]{\text{Luttinger}} \mathrm{FS}(X_2/K_{X_2}). \tag{31}$$

Conjecture 7.27.

$$Don(\mathrm{FS}(X_2/K_{X_2})) = Don(\mathrm{FS}(X_1/K_{X_1}))\cdot S(F), \tag{32}$$

where S is a function of an additional basic class F.

Here Don are $\mathrm{SU}(n)$ bundles for $n > 2$. This suggests that gaps of Orlov spectra change from $\mathrm{FS}(X_1/K_{X_1})$ to $\mathrm{FS}(X_2/K_{X_2})$.

We also conjecture:

Conjecture 7.28. The creation of additional basic classes creates gaps in dynamical spectra of smooth projective varieties.

This suggests that the dynamical spectrum is a birational invariant.

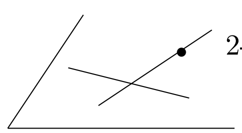
Conjecture 7.19 suggests that we have a correspondence:

$$Don((\text{Ell Surf})\log \mathrm{tr}) = Don(\text{Ell Surf})\Delta(e^t) \tag{33}$$

$$\text{Ell Surf} \xrightarrow{\text{log transforms}} Dolg\ 2,3 \xrightarrow{\text{log transforms}} Dolg\ p,q\cdots \tag{34}$$

Similarly in the case of 3-dim Fano.

$$\mathrm{FS}\left(\begin{array}{c}\mathbb{C}^3\\ w\big\downarrow\\ \mathbb{C}^2\end{array}\right) = \mathrm{D^b}(\mathbb{P}^3)$$



2-dim log transform$= \mathrm{D^b}(X_3)$

We have

$$Don(\mathrm{D^b}(X_3)) = Don(\mathrm{D^b}(\mathbb{P}^3))\cdot \Delta_1(e^{t_1})\cdot \Delta_2(e^{t_2}). \tag{35}$$

So we have a 3-dim analogue:

$$\mathbb{P}^3 \xrightarrow{\text{log transform}} X_1 \xrightarrow{\text{log transform}} X_2. \tag{36}$$

Conjecture 7.29. The categorical Donaldson invariants of 3-dimensional Fanos are connected by the formula

$$Don(X_1) = Don(X_2)\Delta_1(e^{t_1})\Delta_2(e^{t_2}), \tag{37}$$

where Δ_1 and Δ_2 correspond to 2 log transforms.

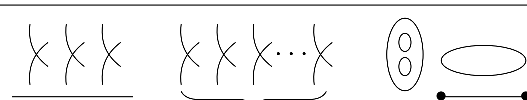
Similarly the same should be true for 4-dimensional Fanos.

Elliptic Surfaces	$Don(X) = Don()\Delta(e^{t})$	t is not a blow-up basic class
3-dim Fanos	$Don(X) = Don\ \Delta_1(e^{t_1})\Delta_2(e^{t_2})$	t_1, t_2 are not blow-up classes
4-dim Fanos	$Don(X) = Don\ \Delta_1(e^{t_1})\Delta_2(e^{t_2})\Delta_3(e^{t_3})$	t_1, t_2, t_3 are not blow-up classes

This leads to a parallel between the theory of elliptic fibrations and Fano 3-folds of $Pic = 1$.

Elliptic Fibrations	3-dim Fanos
$E_1 \xrightarrow[\text{surgeries}]{\text{rational blow-ups}} E_2$ Log transforms rational blow-downs	$X_1 \overset{\text{degeneration}}{\rightsquigarrow} X'_{\text{tor}}$ $X'_{\text{tor}} \xmapsto{\text{bir}} X''_{\text{tor}}$ $X_2 \overset{\text{degeneration}}{\rightsquigarrow} X''_{\text{tor}}$

In the case of Elliptic surfaces, the Donaldson invariants depend on the vanishing classes of blow-downs. Indeed we start with $\mathbb{P}^2$ blown up in 9 points - intersection of 2 smooth cubics in $\mathbb{P}^2$

Surfaces	$\mathbb{P}^2 \dashleftarrow \underset{t_1,\dots,t_9}{\widehat{\mathbb{P}^2}_{p_1,\dots,p_9}} \xrightarrow[\text{Log 3}]{\text{Log 2}}$ Dolg 2,3
Sheaf of Categories	 12 9 basic classes spectral network new basic class

and end with one basic class which is represented by a spectral network. See [19].

Consider the degeneration and regeneration of 3-dimensional Fanos.

$$X_1 \xrightarrow{deg} X'_{\text{tor}} \xrightarrow{bir} X'_{\text{tor}} \xleftarrow{bir} X_2 \tag{38}$$

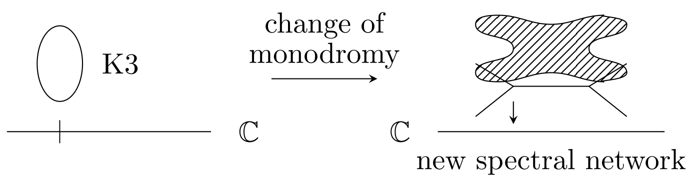
Conjecture 7.30. The degeneration of MHS associated with degenerations $X_1 \xrightarrow{deg} X'_{\text{tor}}$, $X_2 \xrightarrow{deg} X''_{\text{tor}}$ of 3-dimensional Fanos leads to a connection $Don(X_1) = Don(X_2)$ $(Don(\text{MHS}))$.

Conjecture 7.31. For 3-dim Fanos X_1, X_2, $\mathrm{Pic}(X_1) = \mathrm{Pic}(X_2)$,

$$Don(\mathrm{MHS})$$

is connected with the Bardelli invariant $\psi(\alpha_1, \alpha_2)$. (See [3]).

Idea of Proof. As before basic classes are obtained via change of sheaf category construction.



Change of monodromy leads to

- new spectral network.
- new moduli space of a new basic class.
- change of Bardelli form.

□

We need to mention that there is an issue with wall crossing. In order to compare Donaldson invariants we need wall crossing formulas for moduli spaces of objects with wall crossing formulas in Donaldson theory.

Conjecture 7.32. In the case of Dolgachev surface we have a wall crossing formula on the Donaldson invariants for moduli spaces of objects which in limit produces the classical Donaldson formula $Don(C) = Don(C') + \Sigma_i e_i$.

In dim 4 we have also (4-dim cubic)

$$\mathrm{LG}(X_1) \xrightarrow{\text{log transform}} \mathrm{LG}(X_2). \tag{39}$$

This log transform changes the basic classes. In the case of X_1 being rational 4-dim cubic and X_2 being generic 4-dim cubic, we lose $\mathcal{O}_{\mathrm{K3}}$ as a basic class (see [21] and [14]).

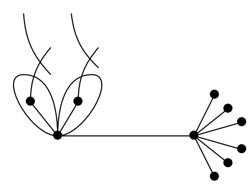
We arrive at several correspondences with classical Donaldson theory.

Classical Donaldson Theory	Categorical Donaldson Theory
$X = X_1 \#_C X_2$	sheaf of categories is not connected with a potential
	local sheaf of categories does not split in two sheaves which are not connected with potentials

We recall two examples.

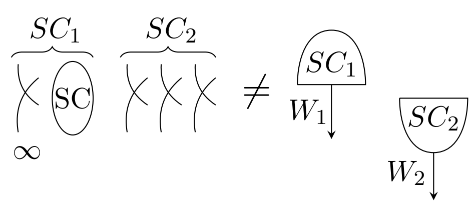
1. Dolgachev surface.

$$X = X_1 \# X_2$$



No function on these sheaves of categories.

2. 4-dim cubic.



Obstructions to such splitting are the moduli spaces of stable objects.

The following theorem could lead to a new approach to irrationality.

Theorem 7.33. *Let X be a rational manifold. Then $SC(X)$ - the sheaf of categories associated with the LG mirror of X can be connected with a potential.*

We conjecture that the sheaves of categories $SC(X)$ are determined by moduli spaces of objects.

Conjecture 7.34. The sheaves of categories $SC(X)$ are determined by moduli spaces of objects and sheaves on dg-algebras on them.

The Donaldson invariants (if properly defined) determine if $SC(X)$ are sheaves of categories given by a function.

Clearly constructing sheaves of categories not coming from potentials could have an important application in studying rationality questions. Before giving several procedures of constructing sheaves of categories $SC(X)$ not connected with potentials we return to our basic example - the 2-dimensional LG model for Dolgachev surface.

Theorem 7.35. *The sheaves of categories $SC(X)$ associated with the 2-dimensional LG model for Dolgachev surface is not connected with a potential.*

Indeed in this case the sheaf of categories does not come form the push-forward of the structure but from a gerb. The 4-dimensional LG model for Dolgachev surface produces a sheaf of categories which is connected with a potential. This observation suggests the following:

Theorem 7.36. *The following procedure could lead to sheaves of categories $SC(X)$ not connected with a potential:*

1. *Changing the structure sheaf of the initial sheaf of categories coming from a potential to a gerb.*

2. *Taking a covering of a sheaf of categories.*

3. *Taking a part of a sheaf of categories.*

This observation suggests that sheaves of categories coming from potentials play the role of simple 4-manifolds in Donaldson's theory and the procedure of splitting these sheaves is the analogue of the procedure of stretching the neck.

Question 7.37. Consider $SC(X)$ - the sheaf of categories of LG model of a Fano threefold X. Assume that $SC(X) = (SC_1, SC_2)$ i.e. we split $SC(X)$ to two sheaves of categories and one of them SC_2 is not connected with a potential. Can we then claim that X is not rational?

A positive answer to this question will build a categorical parallel to Voisin's theory of CH^0- trivial Fano varieties.

The splitting SC_1, SC_2 as sheaves of categories can be interpreted as limited stability conditions and as testing configurations. These are more general than just to split LG models - which produces more opportunities for applying Voisin's techniques.

7.2 More on sheaves of categories.

In this subsection we will try to summarize our observations.

Definition 7.38. Let $f : Y \to \mathbb{C}$ be a regular map. Then $f_*(D_{\mathcal{O}_Y})$ is a **sheaf of categories connected with a function (with a potential)** . (Here $D_{\mathcal{O}_Y}$ is the category of $\mathcal{O}_Y$-D modules.)

We can also consider Fukaya - Seidel category with coefficients in a stack Z - $FS(Y, Z)$.

How to obtain FS$(Y, \mathcal{F})$ not connected with a function:

1. Change $\mathcal{O}_Y$ on a Gerb G (Stack). Example of that is Dolg 2,3, 2-dim LG model.
2. Taking finite group quotients of Y.
3. Degeneration, taking open subset $Y' \subset Y$, regeneration.

This suggests a possible conceptual definition of FS(Y, f, G) in the situation above.

Definition 7.39. FS(Y, f, G) is defined as global sections of the sheaf of categories $f_*(D_{\mathcal{O}_Y})$. or $f_*(D_G)$.

Conjecture 7.40. The change of the sheaf of categories from $f_*(D_{\mathcal{O}_Y})$ to $f_*(D_G)$ leads to the following:

1. FS(Y, f, G) can attain a phantom.
2. FS(Y, f, G) attains additional categorical basic classes.

Similarly we define categories of matrix factorizations with coefficients, MF(Y, f, G).

This means the pair

$$\begin{array}{ccc} P_1 \xrightarrow{d} P_0 & & P_1 \xrightarrow{d} P_0 \\ d^2 = f & \text{changes to} & d^2 = f \\ P_i \text{ is an } \mathcal{O}_Y\text{-module} & & P_i \text{ is a } G\text{-module} \end{array}$$

Definition 7.41. We call $\mathrm{MF}(Y, f, G)$ a **category of matrix factorization with coefficients**.

Conjecture 7.42. The change $\mathrm{MF}(Y, f) \to \mathrm{MF}(Y, f, G)$ leads to new categorical basic classes.

We can say this differently:

$$\begin{array}{l} Y, G \text{ - a gerb} \\ \downarrow f \\ Z \end{array}$$

We get a sheaf over Z, $\mathrm{D^b}(Y, G)/Perf(Y, G)$.

$$\mathrm{MF}(Y, G) := \text{global section of } \mathrm{D^b}(Y, G)/Perf(Y, G). \tag{40}$$

So we summarise the proposed definition of A and B side sheaves of categories connected and not connected with a function.

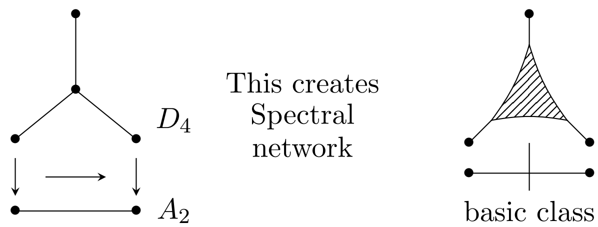
	A	B
Sheaf of categories with a function	$f : Y \to \mathbb{C}$ $\mathrm{FS}(Y, f, f_*\mathcal{O}_Y)$	$f : Y \to \mathbb{C}$ $\mathrm{MF}(Y, f, f_*\mathcal{O}_Y)$
Sheaf of categories with a function	$Y \to Z$-stack G-gerb only $\mathrm{FS}(Y, Z, G)$	$Y \to Z$-stack G-gerb only $\mathrm{MF}(Y, Z, G)$

Example 7.43. Consider Fuk(Dolg 2,3) $\cong \mathrm{MF}(Y, G)$.

$$\begin{tikzcd} \mathrm{Fuk}(\widehat{\mathbb{P}^2}_{p_1,\dots,p_9}) \arrow[r, phantom, "\cong"] \arrow[d, "\mathrm{Log\,tr}", "\not\Downarrow"'] & \mathrm{MF}(\bigoplus_1^{12} \mathbb{C}[x,y]/_{x^2+y^2}) \arrow[d, "\mathrm{Log\,tr}"] \\ \mathrm{Fuk}(\text{Dolg } 2,3) \arrow[r, phantom, "\cong"] \arrow[ur, phantom, "\text{different basic classes}"] & \mathrm{MF}(Y, G) \end{tikzcd}$$

Fuk(Dolg 2,3) has no phantoms since it is a Calabi -Yau category. But it has new basic classes. Similarly new basic classes appear in Fukaya categories after rational blow-down, surgery.

Example 7.44. Consider $A_2 \times A_2 \times A_2$.



We can define $\mathrm{MF}(A^2, D_4)$ with additional basic classes.

All these suggest a generalization of Orlov's Theorem for sheaves of categories with coefficients.

Conjecture 7.45. 1. $\mathrm{MF}(Y, G) =< \mathrm{D^b}(A, G'), E_1, \ldots, E_n >$.

2. $\mathrm{D^b}(B, G') =< \mathrm{MF}(Y, G), E_1, \ldots, E_n >$.

Here A is a category of general type, B is a category of Fano type. G, G' are gerbs (stacks).

Some version of this conjecture appears in [**?**].

Acknowledgement

We are grateful to D. Auroux, S. Donaldson, F. Haiden, P. Horja, M. Kapranov, G. Kerr, M. Kontsevich, A. Kuznetsov, J. Morgan and P. Pandit for the help with the paper. The authors were supported by Simons research grant, NSF DMS 150908, ERC Gemis, DMS-1265230, DMS-1201475 and OISE-1242272 PASI.

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