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Applications of Gauged Gromov-Witten Theory: A Survey

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Abstract

This is a short survey article on applications of gauged Gromov-Witten theory into the understanding of Gromov-Witten theory of GIT quotients of a smooth projective variety by a reductive group. In particular we will explain how several classical results in equivariant cohomology extend to quantum cohomology. These include wall crossing results, Witten localisation and abelianisation. We also describe a GIT version of the so called crepant conjecture.

In memory of Samuel Gitler

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1 Introduction

This note is a survey, and our objective is to explain several results in the series of joint papers [22, 19, 20, 21] with Chris Woodward. This material is based on several talks given by the author, in particular a series of lectures given at CIMPA-CIMAT-SWAGP Research School at the CIMAT, México (Winter 2013) and a lecture given at the Second International School on TQFT, Langlands and Mirror Symmetry, in Playa del Carmen (Spring 2014). The results we present are essentially extensions of well known classical results in equivariant cohomology into quantum cohomology, and as such we will use the term *quantisation* in this sense. In Section 2, we begin by revising classical tools relating the cohomology of a GIT quotient $X /\!\!/ G$ and the equivariant cohomology $H_G(X)$, chiefly to introduce notation. We then discuss Kalkman's wall crossing formula, Martin's abelianisation formula and Witten's localisation, which we are the results we are quantising. These results can be understood in terms of trace maps (integration), the Kirwan map and certain fixed point traces, twisted by Euler classes corresponding to fixed point components. We will see that the same results will hold in quantum cohomology, provided that we appropriately redefine these main components. The traces are replaced by appropriately introducing higher degree maps and defining generating functions or potentials of gauged and usual Gromov-Witten invariants. The Kirwan map is replaced by Woodward's quantum analogue and the fixed point contributions will arise as potentials twisted by Euler classes of index bundles associated to fixed points. Due to the nature of the paper, we will only focus on explaining the results and we will do some basic examples. In Section 3 we discuss the construction of the moduli space of Mundet semi-stable gauged maps and the definition of the gauged Gromov-Witten potential. Here we emphasise that gauged maps are equivariant lifts of maps into $X /\!\!/ G$, and to recover the maps into the quotient one needs to consider limits in the semi-stability condition. In Section 4.1 we proceed to explain wall crossing for gauged Gromov-Witten potentials as well as its descent (adiabatic limit) to Gromov-Witten potentials of $X /\!\!/ G$. We then explain in Section 4.4 how this can be used to show that the Gromov-Witten potentials of two GIT quotients related by a crepant birrational map are essentially the same. In Section 4.5 we describe how gauged Gromov-Witten theory can be used to quantise Witten's localisation and then use it to show abelianisation formulas, that is, to relate the Gromov-Witten theories of $X /\!\!/ G$ and that of the quotient $X /\!\!/ T$ by its maximal torus.

2 Classical equivariant cohomology results

We follow the same set up as in [34]. Let G be a reductive group and consider a non-singular polarised G variety (X, L), that is an ample line bundle $L \to X$ equipped with a linearisation (lift) of the action. We let X^{ss} denote the semistable locus, that is the subset of points $x \in X$ such that $s(x) \neq 0$ for some invariant section $s \in H^0(X, L^{\otimes n})^G$ and some integer n. We will assume in most cases, unless explicitly stated, that all semi-stable elements are strictly stable (stable=semistable); in such case G acts with finite stabilisers on the semistable set, so that the GIT quotient $X /\!\!/ G := X^{ss}/G$, is a Deligne-Mumford stack. In this paper we will assume that the quotient is actually smooth, to simplify the exposition, however the results hold assuming that the quotient has a projective coarse moduli. Let $H_G(X;\mathbb{Q})$ denote the equivariant cohomology, which we will also identify with the cohomology of the quotient stack X/G. In symplectic geometry, Kempf-Ness [31] identified the GIT quotient as the symplectic quotient: the quotient of the zero-level set of the moment map by the maximal compact in G. Let $\kappa_{X,G}$: $H_G(X;\mathbb{Q}) \to H(X /\!\!/ G)$ denote the *Kirwan map*, given by restriction to the semistable locus and then descent to the quotient. Integration over $X /\!\!/ G$ defines a *trace*

$$(1) \qquad \textbf{(Trace)} \quad \tau_{X/\!\!/G} : H(X /\!\!/ G) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X/\!\!/G]} \alpha.$$

We now review classical equivariant cohomology results.

2.1 Wall crossing and Kalkman's formula

The GIT quotient $X /\!\!/ G$ depends on L, or equivalently, a choice of moment map. The dependency on the variation of L is studied in [17, 7, 12, 41]. Under suitable stable=semistable conditions, $X /\!\!/ G$ undergoes a sequence of weighted blow-ups and blow-downs. The class of birational equivalences which appear via variation of GIT is quite large. In fact, for the so-called *Mori dream spaces*, any birational equivalence can be written as a composition of birational equivalences induced by variation of GIT.

We are interested in what happens at the level of intersection parings in cohomology, and in GW theory correlators. The following material is well known, however we need it to introduce notation to explain our results.

Kalkman [27] studied the question of how the cohomology of the quotient depends on the polarisation, and provides a wall-crossing formula for the intersection pairings. Let $X /\!\!/_\pm G$ denote the associated GIT quotients corresponding to two polarisations $L_\pm \to X$, and let $\kappa^G_{X,\pm} : H_G(X) \to H(X /\!\!/_\pm G)$ be the Kirwan maps and let $\tau_{X/\!\!/_\pm G}$: $H(X /\!\!/_\pm G) \to \mathbb{Q}$ denote integration over $X /\!\!/_\pm G$. Kalkman's formula expresses the difference between the integrals $\tau_{X/\!\!/_\pm G} \circ \kappa^G_{X,\pm}$ as a sum of fixed point contributions for the integral of a class $\alpha \in H_G(X)$ over

X. In other words, it measures the failure of the following square to commute,

$$\begin{tikzcd}
 & H_G(X) \arrow[dl, "\kappa^G_{X,+}"'] \arrow[dr, "\kappa^G_{X,-}"] & \\
H(X /\!\!/_+ G) \arrow[dr, "\tau_{X /\!\!/_+ G}"'] & & H(X /\!\!/_- G) \arrow[dl, "\tau_{X /\!\!/_- G}"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

by an explicit sum of wall-crossing terms. To explain properly what these contributions are, consider the interpolation $L_t := L_-^{(1-t)/2} \otimes L_+^{(1+t)/2}$ for rational $t \in (-1,1)$. The family of GIT quotients $X /\!\!/_{L_t} G$ given by the variation of semi-stability is recovered by the *master space* [41], the quotient $M = \mathbb{P}(L_+ \oplus L_-) /\!\!/ G$. M itself has a $\mathbb{C}^\times$ action given by the scaling on the fibres of M. There is a natural linearisation of this action, induced by $\mathcal{O}(1) \to \mathbb{P}(L_+ \oplus L_-)$, which in turn produces a family of quotients $M /\!\!/_t \mathbb{C}^\times$ by considering the semi-stability with respect to $\mathcal{O}(t)$. The main result of variation of GIT is that under appropriate stable=semi-stable conditions $M /\!\!/_t \mathbb{C}^\times$ is naturally identified with $X /\!\!/_{L_t} G$. The fixed point set of the $\mathbb{C}^\times$-action on M is given as follows. For any $\zeta \in \mathfrak{g}$, we denote by $G_\zeta \subset G$ the stabilizer of ζ under the adjoint action of G. Let $T \subset G$ be a maximal torus and $\mathfrak{t} \subset \mathfrak{g}$ the corresponding Cartan algebra, and $W = N(T)/T$ its Weyl group. For any $\zeta \in \mathfrak{t}$, we denote by W_ζ resp. $W_{\mathbb{C}\zeta}$ the group of $w \in W$ that fix the element $\zeta \in \mathfrak{g}$ resp. line $\mathbb{C}\zeta \subset \mathfrak{g}$. Thus the quotient $W_{\mathbb{C}\zeta}/W_\zeta$ is either isomorphic to $\{\pm 1\}$ or to $\{1\}$, depending on whether or not there is a Weyl group element acting as -1 on $\mathbb{C}\zeta$. Suppose that stable=semistable for the G-action on $\mathbb{P}(L_- \oplus L_+)$, so that M is a smooth proper Deligne-Mumford stack with $\mathbb{C}^\times$ action. Any $x \in M^{\mathbb{C}^\times}$ with $x = [l]$ for some $l \in \mathbb{P}(L_- \oplus L_+)$ has the property that for all $z \in \mathbb{C}^\times$, $zl = z^\zeta l$ for a unique $\zeta \in \mathfrak{g}$. For each $\zeta \in \mathfrak{t}$ there is a morphism $\iota_\zeta : X^\zeta /\!\!/_t (G_\zeta/\mathbb{C}^\times_\zeta) \to M^{\mathbb{C}^\times}$ with fiber $W_{\mathbb{C}\zeta}/W_\zeta$. The images of ι_ζ cover $M^{\mathbb{C}^\times}$, disjointly after passing to equivalence classes of one-parameter subgroups. For any $\alpha \in H_G(X)$, the pull-back of $\widetilde{\kappa}(\alpha)|_{M^{\mathbb{C}^\times}}$ under ι_ζ is equal to image of α under the restriction map

$$H_G(X) \to H_{\mathbb{C}^\times_\zeta}(X^\zeta /\!\!/_t (G_\zeta/\mathbb{C}^\times_\zeta)).$$

The pull-back of the normal bundle $N_{M^{\mathbb{C}^\times}}$ of $M^{\mathbb{C}^\times}$ under ι_ζ is canonically isomorphic to the image of $N_{X^\zeta}/(\mathfrak{g}/\mathbb{R}\zeta)$ under the quotient map

$X^\zeta \to X^\zeta /\!\!/ (G_\zeta/\mathbb{C}_\zeta^\times)$, by an isomorphism that intertwines the action of $\mathbb{C}_\zeta^\times$ on $(N_{X^\zeta}/(\mathfrak{g}/\mathbb{R}\zeta)) /\!\!/ (G_\zeta/\mathbb{C}_\zeta^\times)$ with the action of $\mathbb{C}^\times$ on $N_{M^{\mathbb{C}^\times}}$. The group $\mathbb{C}_\zeta^\times \subset G_\zeta$ acts trivially on X^ζ, which is therefore a $G_\zeta/\mathbb{C}_\zeta^\times$-space. This is the "smaller structure group" acting on the wall terms. For any fixed point component $X^{\zeta,t} \subset X^\zeta$ that is t-semistable, we denote by $\nu_{X^{\zeta,t}}$ the normal bundle of $X^{\zeta,t}$ modulo $\mathfrak{g}/\mathbb{C}\zeta$, quotiented by $G_\zeta/\mathbb{C}_\zeta^\times$. Let $j_{\zeta,t} : H_G(X) \to H_{\mathbb{C}_\zeta^\times}(X^{\zeta,t}) \to H(X^{\zeta,t} /\!\!/ (G_\zeta/\mathbb{C}_\zeta^\times))$, then we define

(2) **(Fixed Point Trace)** $\tau^{G_\zeta}_{X,\zeta,t} : H_G(X) \to \mathbb{Q}[\xi, \xi^{-1}]$,

$$\alpha \mapsto \int_{[X^{\zeta,t} /\!\!/ (G_\zeta/\mathbb{C}_\zeta^\times)]} j_{\zeta,t}(\alpha) \cup \mathrm{Eul}_{\mathbb{C}_\zeta^\times}(\nu_{X^{\zeta,t}})^{-1}$$

where ξ is the equivariant parameter for $\mathbb{C}_\zeta^\times$. Therefore, with the considerations above (stable=semistable for the G action on $\mathbb{P}(L_- \oplus L_+)$) we have

(3) **(Kalkman's wall-crossing formula)** $\tau_{X/\!\!/_+ G} \circ \kappa^G_{X,+} -$

$$\tau_{X/\!\!/_- G} \circ \kappa^G_{X,-} = \sum_{t \in (-1,1), [\zeta]} \frac{|W_\zeta|}{|W_{\mathbb{C}\zeta}|} \mathrm{Resid}_\xi \, \tau^{G_\zeta}_{X,\zeta,t},$$

where the sum is over one-parameter subgroups $[\zeta]$ of G, up to conjugacy. The formula (3) holds more generally, e.g. for certain quasi-projective varieties, such as vector spaces whose weights are contained in an open half-space.

Example 2.1.1. Let us exemplify the notation above. Let $G = \mathbb{C}^\times$ act on $X = \mathbb{C}^N$ by scalar multiplication, so that $H_G(X) = \mathbb{Q}[\xi]$. Let $L_\pm$ correspond to the weights ± 1, thus $X /\!\!/_- G$ is empty and $X /\!\!/_+ G = \mathbb{P}^{N-1}$. There is a unique singular value $t = 0$, corresponding to the origin $0 \in X$. $\kappa^G_X : H_G(X) \to H(X /\!\!/ G)$ sends $\xi \in H^2_G(X)$ to the hyperplane class $h \in H^2(X /\!\!/ G)$. The integrals $\int_{\mathbb{P}^{N-1}} h^a$ for $a \in \mathbb{Z}_{\geq 0}$ can be computed via wall-crossing. For the empty side, the integral is zero. By the Kalkman formula (3)

$$\int_{\mathbb{P}^{N-1}} h^a = \mathrm{Res}_\xi \int_{[0]} \xi^a \cup \mathrm{Eul}_G(\mathbb{C}^N)^{-1} = \mathrm{Res}_\xi \, \xi^a/\xi^N = \begin{cases} 1 & a = N-1 \\ 0 & \text{otherwise} \end{cases}$$

showing that h^{N-1} is the dual of the fundamental class.

2.2 Witten localisation and Abelianisation

It is natural to ask if we can compute the composition $\tau_{X/\!/G} \circ \kappa_{X,G}$ exclusively in terms of the G-equivariant cohomology on X, for instance, to compute the cohomology of $X /\!/ G$ in terms of the G-equivariant cohomology of X. Witten [42] introduced *non-abelian localisation* to compute $\tau_{X/\!/G} \circ \kappa_{X,G}$ in terms of a trace map

(4) **(Witten's Trace)** $\tau_X^G : H_G(X) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X]\times\mathfrak{k}} \alpha,$

which is given by integration over X *and* a unitary form $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$. The integral of a polynomial over $\mathfrak{k}$ may be defined via various regularisation procedures, see [38, 37, 44], which we will not explain, since they are unnecessary for the quantised version in Section 4.2.1. Our original motivation for the quantum version of Witten's localisation was the *quantum Martin conjecture* of Bertram et al [4] which compares Gromov-Witten invariants of a GIT $X /\!/ G$ and the quotient $X /\!/ T$ by a maximal torus $T \subset G$, as we will see in Section 4.5. Let us first discuss classical abelianisation or *Martin's formula [33]*. T acts on (X, L) as well, and it is not hard to see that the semistable locus of the G action $X^{ss}(G)$ lies in $X^{ss}(T)$ so that there is a natural map $X^{ss}(G)/T \to X^{ss}(T)/T = X /\!/ T$ and a quotient $X^{ss}(G)/T \to X^{ss}(G)/G = X /\!/ G$ by the residual action. The relation of the two GIT quotients is given by $\nu_{\mathfrak{g}/\mathfrak{t}}$, the bundle over $X /\!/ T$ induced from the trivial bundle with fibre $\mathfrak{g}/\mathfrak{t}$ over X. We let $\tau_{X/\!/T} : H(X /\!/ T) \to \mathbb{Q}$ denote the $\mathrm{Eul}(\nu_{\mathfrak{g}/\mathfrak{t}})$-twisted integration map,

$$\tau_{X/\!/T} : H(X /\!/ T) \to \mathbb{Q}, \quad \alpha \mapsto \int_{[X/\!/T]} \alpha \cup \mathrm{Eul}(\nu_{\mathfrak{g}/\mathfrak{t}}).$$

Let $W = N(T)/T$ denote the Weyl group of $T \subset G$ and $\mathrm{Restr}_T^G : H_G(X) \to H_T(X)$ the restriction map, which induces an isomorphism $H_G(X) \to H_T(X)^W$. Suppose that stable=semistable for the actions of T and G on X. Then integration over $X /\!/ G$ and $X /\!/ T$ are related by

(5) **(Martin's Abelianisation formula)** $\tau_{X/\!/G} \circ \kappa_{X,G} =$

$$|W|^{-1} \tau_{X/\!/T} \circ \kappa_{X,T} \circ \mathrm{Restr}_T^G .$$

In other words, the following diagram commutes:

$$\begin{tikzcd} & H_G(X,\mathbb{Q}) \cong H_T(X,\mathbb{Q})^W \arrow[dl, "\kappa_{X,G}"'] \arrow[dr, "\kappa_{X,T}"] & \\ H(X /\!\!/ G,\mathbb{Q}) \arrow[dr, "\tau_{X/\!\!/G}"'] & & H(X /\!\!/ T,\mathbb{Q}) \arrow[dl, "|W|^{-1}\tau_{X/\!\!/T}"] \\ & \mathbb{Q} & \end{tikzcd}$$

Moreover Martin proved that there exists a surjective map

$$\mu_T^G : H(X /\!\!/ T) \to H(X /\!\!/ G) \tag{6}$$

whose kernel is the cup product with $\mathrm{Eul}(\mathfrak{g}/\mathfrak{t})$. Given classes $\alpha_G \in H(X /\!\!/ G)$, $\alpha_T \in H(X /\!\!/ T)$, we have $\alpha_G = \mu_T^G \circ \alpha_T$ iff

$$\int_{[X/\!\!/G]} \alpha_G \cup \kappa_{X,G}(\beta) = |W|^{-1} \int_{[X/\!\!/T]} \alpha_T \cup \kappa_{X,T}(\beta) \cup \mathrm{Eul}(\mathfrak{g}/\mathfrak{t})$$

for all $\beta \in H_G(X) \cong H_T(X)^W$.

3 Gauged Gromov-Witten theory

In order to extend the results explained in the previous section, we first need to introduce gauged maps as equivariant lifts of stable maps. Let C be a smooth connected projective curve. Since $X /\!\!/ G$ is embedded in the quotient stack X/G, a map $C \to X /\!\!/ G$ is in principle a *gauged map*: a morphism $C \to X/G$, or more explicitly, a pair (P, u) of a principal G-bundle $P \to C$ over C and a section u of the associated bundle $P(X) := P \times_G X \to C$. In this way, gauged maps are naturally the algebraic analogues of *symplectic vortices* [8, 9, 8, 18] similar to the gauged σ-models of Witten [43]. Therefore the space of gauged maps is a natural extension of both, the space of maps $\mathrm{Maps}(C, X)$ and the stack $\mathfrak{B}G$ of principal bundles. Since we are interested in invariants, we will also consider the moduli of maps with n different markings, $c = (c_1, \cdots, c_n) \in C^n$. We consider the stack of n-pointed nodal gauged maps $\overline{\mathfrak{M}}_n^G(C, X)$. These are pairs (u, c) of a stable n-pointed map u :

$\widehat{C} \to C \times X/G$ such that the projection onto the first factor C has class $[C]$. The *degree* of a gauged map is its push forward in $H_2(X/G) = H_2^G(X)$. Here, we identify the (co)homology of the quotient stack X/G with the equivariant (co)homology of X [13]. Thus for a gauged map, the principal bundle lies exclusively on a principal component $\widehat{C}_0$ of the nodal curve $\widehat{C}$, isomorphic to C, and the principal bundle over all other components of $\widehat{C}$ map to a point. This means that the nodal map defines a stable map $u : \widehat{C} \to P(X)$ with base class $[C]$, that is, the composition of u with the projection $P(X) \to C$ has class $[C]$. This means that nodal gauged maps are allowed to acquire "fibre bubbles", rational bubble trees attached on the fibres. The underlying curve in the gauged map lies in the Fulton-McPherson compactification $\overline{\mathcal{M}}_n(C)$ of the space of maps, rather than the Deligne-Mumford compactification as in usual GW theory, since it has a principal component. This compactification is the space $\overline{\mathcal{M}}_{g(C),n}(C, [C])$ of genus $g(C)$ stable maps into C, of fixed degree $[C] \in H_2(C)$. We let $\overline{\mathfrak{M}}_n^{G,st}(C, X)$ denote the substack of gauged maps for which $(u, c) : \widehat{C} \to P(X)$ is a stable map in the usual sense of Kontsevich. In general $\overline{\mathfrak{M}}_n^{G}(C, X), \overline{\mathfrak{M}}_n^{G,st}(C, X), \mathfrak{M}_n^G(C, X)$ are non-finite type, non-separated Artin stacks [45, Theorem 5.2]. Since we are interested in invariants, we need to impose a semi-stability condition to guarantee a good moduli space supporting a perfect obstruction theory. We want to differentiate the construction above with equivariant GW theory, as introduced by Givental [15]. In this case, the stable maps are requested to take values in the *fibres* of the homotopy quotient $X \times_G EG \to BG$ (the geometric realisation of the stack X/G), that is, they only lie in the X direction.

3.1 Mundet semi-stability and gauged GW invariants

A choice of polarisation $L \to X$ of X (equipped with a compatible lift of the G-action on L) gives a semi-stability condition for gauged maps, as introduced by Mundet i Riera [35, 36]. Mundet's semi-stability couples Ramanathan's semi-stability for principal bundles and the Hilbert-Mumford semi-stability for X. Suppose $(P, u) : C \to X/G$ is a gauged map. Let R denote a parabolic subgroup of G, and let $\sigma : C \to P/R$ denote a reduction of structure group. Let λ be a co-weight of R (we identify $\mathfrak{g}\cong\mathfrak{g}^\vee$) and consider the one-parameter subgroup $\mathbb{C}^\times \to R$ given by $z \mapsto z^{-\lambda} = \exp(-\ln(z)\lambda)$. The pair (λ, σ) yields the *associated graded*, a pair $(\mathrm{Gr}(P), \mathrm{Gr}(u))$ consisting of a bundle $\mathrm{Gr}(P)$ whose structure group reduces to the Levi subgroup of P and a stable section $\mathrm{Gr}(u) : \widehat{C} \to$

$\mathrm{Gr}(P)(X)$ from a nodal curve $\widehat{C}$. If $R = LU$ is a Levy decomposition, then $\mathrm{Gr}(P) = i_* p_* \sigma^* P$, where $p : R \to R/U = L, i : L \to G$ are the natural maps. For our purposes we prefer the presentation via degeneration which we will describe in what follows. For each $z \in \mathbb{C}^\times$, we consider the G bundle $P_{(z,\sigma)} = \sigma^* P \times_{R, z^{-\lambda}} G$ where the action on G is by conjugation. The limit as $z \to 0$ exists, if λ is *dominant*, and agrees with $\mathrm{Gr}(P)$. By λ dominant we mean that it is zero on the connected component of the centre of G and it is positive on the roots of T. The twisted map $z^{-\lambda} u$ is a section of the associated bundle $P_{(z,\sigma)}(X)$, and its limit as $z \to 0$ converges in the Gromov sense to a nodal section of $\mathrm{Gr}(P)(X)$. This is the graded section $\mathrm{Gr}(u)$. Over the principal component $C_0 \cong \widehat{C}$, $\mathrm{Gr}(u)$ takes values in the fixed point set X^λ of the automorphism induced by λ, and so has a well-defined *Hilbert-Mumford weight* $\mu_{HM}(\sigma, \lambda)$ determined by the polarisation L, given as the usual Hilbert-Mumford weight [34, Section 2] at a generic point in C_0. The *Ramanathan weight* (cf. [2, Definition 3.2], [39, 40]) $\mu_R(\sigma, \lambda)$ of (P, u), with respect to (σ, λ) is given by the first Chern number of the line bundle determined by λ: $\mu_R(\sigma, \lambda) = \int_{[C_0]} c_1(\sigma^* P \times_R \mathbb{C}_\lambda)$.

Definition 3.1.1. The *Mundet weight* is then the coupling

$$\mu_M(\sigma, \lambda) := \mu_{HM}(\sigma, \lambda) + \mu_R(\sigma, \lambda). \tag{7}$$

The map (P, u) is *Mundet semistable* if $\mu_M(\sigma, \lambda) \leq 0$ for all pairs (σ, λ) for λ dominant, and *Mundet stable* if the above inequalities are satisfied strictly. This definition carries to stable gauged maps, since there is no contribution to the weight coming from the bubbles. We denote by $\overline{\mathcal{M}}_n^G(C, X)$ the space of semi-stable nodal pointed gauged maps.

Remark 3.1.2. The Hilbert-Mumford weight can be computed in terms of the Moment map, using the usual correspondence between linearisations of actions and moment maps [28]. Let $\rho : X \to \mathbb{P}(V), V = H^0(X, L)^\vee$ the embedding given by the (very ample) polarisation L, so that $G \hookrightarrow GL(V)$. Let $\Phi = \rho \circ \Phi_{\mathbb{P}} : X \to \mathbb{P}(V) \to \mathfrak{g}_{\mathbb{R}}$ denote the restriction to X of the moment map $\Phi_{\mathbb{P}}$ of $\mathbb{P}(V)$. In this case the Hilbert-Mumford weight is just given by

$$\mu_{HM}(\sigma, \lambda) = \int_{C_0} \langle P(\Phi) \circ \mathrm{Gr}(u), \lambda \rangle, \tag{8}$$

where $P(\Phi)$ is the map Φ induces on $P(X)$, and the integral is over the principal component.

Example 3.1.3. So far the definition of semi-stability was done in the projective setting. If X is affine, we can still define semi-stability in the same exact way, but only for the pairs (σ, λ) for which the graded $(\mathrm{Gr}(P), \mathrm{Gr}(u))$ exists. To exemplify this, consider the diagonal action of $G = \mathbb{C}^\times$ in the affine space $X = \mathbb{C}^N$, where we take the moment map to be $\Phi : \mathbb{C}^N \to \mathfrak{g}_\mathbb{R}$ given by $z \mapsto -|z|^2 + 1$. Let $C = \mathbb{P}$ and $P \to \mathbb{P}$ be a bundle of degree $d \in \mathbb{Z}$. Then, $R = G$ and let $\lambda \in \mathbb{Z}$, denote a coweight of R. Since all the reductions σ are trivial we have $\mathrm{Gr}(P) = P$. The associated bundle $P(X) = P \times_{\mathbb{C}^\times} \mathbb{C}^N = \mathcal{O}(d)^N \to \mathbb{P}$ has sections only if $d \geq 0$. We need to determine which sections $u \in H^0(\mathbb{P}, \mathcal{O}(d)^N)$ are semi-stable. For all $\lambda \in \mathbb{Z}$ the limit $\mathrm{Gr}(u) = \lim_{z \to 0} z^{-\lambda} u$ exists and is zero if $u \equiv 0$; it does not exist if $\lambda > 0$ and $u \neq 0$; and it exists and is zero if $\lambda < 0$ and $u \neq 0$. Note that since X is affine, there is no fibre bubbling, so $\mathrm{Gr}(u) \equiv 0$, whenever the limit above exists. Thus, the weight (7) is given by

$$\mu_M(\sigma, \lambda) = \langle d, \lambda \rangle + \int_\mathbb{P} \langle \Phi(\mathrm{Gr}(u)), \lambda \rangle \, \mathrm{Vol}_\mathbb{P} = \langle d, \lambda \rangle + \lambda,$$

using that $\Phi(\mathrm{Gr}(u)) = 1$ and normalising so that $\mathrm{Vol}(\mathbb{P}) = 1$. Therefore, the semi-stable sections are those non-zero, and the unstable one is $u \equiv 0$. Therefore

$$\mathcal{M}^G(C, X) = (H^0(\mathbb{P}, \mathcal{O}(d)^N) \setminus \{0\})/\mathbb{C}^\times = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}(d)^N)) = \mathbb{P}^{N(d+1)-1}.$$

Since there is no bubbling,

$$\overline{\mathcal{M}}_n^G(C, X) = \mathcal{M}^G(C, X) \times \overline{\mathcal{M}}_n(C) = \mathbb{P}^{N(d+1)-1} \times \overline{\mathcal{M}}_n(C).$$

Example 3.1.4. We can generalise the previous example when $C = \mathbb{P}$. Consider the toric action of a torus $G = T = (\mathbb{C}^\times)^k$ on the affine space $\mathbb{C}^N$ with the weights $\mu_1, \ldots, \mu_N \in \mathfrak{t}^\vee$. The semi-stability condition is determined by a choice of character $\chi \in \mathfrak{t}^\vee$, giving a linearisation on the trivial bundle L_χ. In this case the semistable set is given by $X^{\mathrm{ss}} = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : \chi \in \mathrm{span}\{\mu_i, z_i \neq 0\}\}$, and $X /\!\!/_\chi T$ is a toric variety. The moduli space $\overline{\mathcal{M}}^T(\mathbb{P}, X, d)$ can be computed as follows. A bundle $\mathbb{P}$ with degree $d \in H_2^T(X) = \mathfrak{t}_\mathbb{Z}$ has an associated bundle $P \times_T X$ which is just $\oplus_i \mathcal{O}_\mathbb{P}(d_i)$, where $d_i = \langle d, \mu_i \rangle \in \mathbb{Z}$ and thus $H^0(P \times_T X)$ is identified with $X(d) := \oplus_i \mathbb{C}^{\max(0, d_i+1)}$, where T acts with weight μ_i repeated $\max(0, d_i+1)$. By rescaling the linearisation L_χ^t, and assuming t large enough so that the Mundet weight (7) is dominated by the Hilbert-Mumford term, we obtain $M^T(\mathbb{P}, X, d) = X(d) /\!\!/_\chi T$ which is itself a toric variety. These spaces are the so called quasi-maps as it was defined by Givental in [15].

Fix a degree class $d \in H_2^G(X)$, and assume that stable=semistable for all gauged maps of degree d, and denote the substack of degree d by $\overline{\mathcal{M}}_n^G(C, X, d)$. Similar to GW theory, there is an evaluation at the markings taking values in the quotient stack, $\mathrm{ev} : \overline{\mathcal{M}}_n^G(C, X, d) \to (X/G)^n$, and a forgetful morphism $ft : \overline{\mathcal{M}}_n^G(C, X, d) \to \overline{\mathcal{M}}_n(C)$. The following result is a summary of [19, Proposition 2.1.3], [46, Proposition 5.12, Theorem 5.14, Example 6.6].

Theorem 3.1.5. *Assume stable=semistable for gauged maps, then the moduli stack $\overline{\mathcal{M}}_n^G(C, X, d)$ is a proper, separated Deligne-Mumford stack of finite type with a perfect obstruction theory.*

The properness portion of the result uses a Hitchin-Kobayashi correspondence with the moduli of vortices, see [46, Section 5] for a discussion as well as relations to other compactifications. The perfect obstruction theory is similar to the construction in GW theory. There are universal maps $p : \overline{\mathcal{U}}_n^G(C, X) \to \overline{\mathcal{M}}_n^G(C, X), \mathrm{ev} : \overline{\mathcal{U}}_n^G(C, X) \to X/G$. And thus we can take the complex $Rp_* \mathrm{ev}^* T(X/G)^\vee$ as the perfect obstruction theory, yielding a virtual fundamental class [5]. Define *gauged GW* invariants as

$$\langle \cdot ; \cdot \rangle_{n,d}^G : H_G(X)^{\otimes n} \otimes H(\overline{\mathcal{M}}_n(C)) \to \mathbb{Q} \; ;$$

$$\langle \alpha, \beta \rangle_{g,n}^d := \int_{[\overline{\mathcal{M}}_n^G(C,X,d)]^{\mathrm{vir}}} \mathrm{ev}^* \alpha \cup ft^* \beta,$$

by using integration over the virtual fundamental class.

Example 3.1.6. Let us continue Example 3.1.3 above, with $G = \mathbb{C}^\times$, $C = \mathbb{P}$, $X = \mathbb{C}^N$ so that $X /\!\!/ G = \mathbb{P}^{N-1}$. In this case $H^*(X/G) = \mathbb{Q}[\xi]$, where ξ is the equivariant parameter. Let $n = 3$, be the number of markings. It is not hard to see that the pull back of ξ under (any of) the evaluation map is the hyperplane class $h \in H^2(\mathbb{P}^{N(d+1)-1})$. Then $\overline{\mathcal{M}}_3(C, X, d) = \mathbb{P}^{N(d+1)-1} \times \overline{\mathcal{M}}_3(\mathbb{P})$. The space $\overline{\mathcal{M}}_3(\mathbb{P})$ is six dimensional, and we take $\beta \in H^6(\overline{\mathcal{M}}_3(\mathbb{P}))$ the point class, which corresponds to fixing the markings. Thus

$$\langle \xi^a, \xi^b, \xi^c; \beta \rangle_{n,d}^G = \int_{\mathbb{P}^{N(d+1)-1}} h^a h^b h^c = \begin{cases} 1 & \text{if } a + b + c = N(d+1) - 1; \\ 0 & \text{otherwise.} \end{cases}$$

Note that this does not quite agree with the 3-point GW invariant $\langle h^a, h^b, h^c\rangle_d^{\mathbb{P}^{N-1}}$. For instance if $N = 2$ the Gromov-Witten invariant $\langle h^a, h^b, h^c\rangle_d^{\mathbb{P}^1} = 1$ only if $d = 1$ and $a+b+c = 3$ or $d = 0$ and $a+b+c = 1$. Here we also denote by h the hyperplane class $h \in H^2(\mathbb{P}^1)$, which is the image of the generator ξ under the classical Kirwan map. This discrepancy is corrected of the quantum Kirwan map.

Remark 3.1.7. As a side remark, recall that GW invariants define a cohomological field theory (CohFT) in the sense introduced by Kontsevich and Manin [30]. In a similar fashion gauged GW invariants define a CohFT *trace*, since its underlying combinatorial structure is controlled by parametrised curves, the Fulton-McPherson compactification of curves. We will not need this here, and the interested reader can consult [45, Section 2.2] for more details.

Let

$$(9)\quad \Lambda_X^G := \left\{ \sum_{d \in H_2^G(X;\mathbb{Q})} c_d q^d : \forall e > 0 \ \ \# \left\{ d : \left\langle c_1^G(L), d \right\rangle < e \ \right\} < \infty \right\};$$

$$QH_G(X) = H_G(X, \mathbb{Q}) \otimes \Lambda_X^G$$

denote the *equivariant Novikov ring*, and the G-equivariant quantum cohomology of X respectively, equipped with the usual quantum product using the G-equivariant GW invariants of X taking values in $H(BG)$. Such invariants are defined as the usual invariants, but by equivariant integration over the moduli of maps with respect to the inherited action of G [15]. For the applications we want to discuss, it is convenient to introduce the *gauged potential* as the map

$$(10)\qquad \textbf{(Gauged GW potential)} \qquad \tau_{X,L}^G : H_G(X) \to \Lambda_X^G;$$

$$\tau_{X,L}^G(\alpha) := \sum_{n \geq 0} \sum_{d \in H_2^G(X;\mathbb{Q})} \langle \alpha, \ldots, \alpha; 1\rangle_{n,d}^G \frac{q^d}{n!},$$

and extend to $QH_G(X)$ by linearity. We drop L from the notation whenever there is no risk of confusion.

4 Quantised Results

We are now ready to discuss the main applications.

4.1 Wall crossing for gauged maps.

Similar to the considerations in Section 2.1, we assume that X is a projective G-variety with two given polarisations $L_\pm \to X$. The variation of semi-stability for gauged maps with respect to the interpolation $L_t = L_+^{(t+1)/2} \otimes L_-^{(t-1)/2}$, $t \in (-1,1) \cap \mathbb{Q}$ yields the family of moduli spaces $\overline{\mathcal{M}}_n^G(C, X, d, L_t)$ of L_t-semistable gauged maps. The relation between the associated gauged potentials is given by a formula similar to the classical wall-crossing Equation (3). We follow a similar proof as well. We construct a master space $\tilde{\mathcal{M}}$ equipped with a $\mathbb{C}^\times$ action, realising each $\overline{\mathcal{M}}_n^G(C, X, d, L_t)$ as a quotient $\tilde{\mathcal{M}} /\!\!/_t \mathbb{C}^\times$. Then the result will follow by applying virtual localisation [23] to the master space. Virtual Kalkman in this context is due to Kiem-Li [29], under the same hypotheses imposed in [23], i.e. one needs to verify that the stacks $\overline{\mathcal{M}}_n^G(C, X, d, L_\pm)$ are embedded in *smooth* Deligne-Mumford stacks. For more details the reader can consult [21, Theorem 2.6] and the references therein. From this discussion we have.

Theorem 4.1.1 (Theorem 3.8 [21])**.** *The difference of the gauged potentials is given by*

(11)

(Wall crossing for gauged potentials) $\tau_{X,+}^G(\alpha) - \tau_{X,-}^G(\alpha) =$

$$\sum_\zeta \operatorname{Res} \ \tau_{X,\zeta,L^t}(\alpha).$$

Here the sum of contributions on the right range over one-parameter subgroups, and the terms are residues of the *fixed point* potentials τ_{X,G_ζ,L^t} corresponding to the subgroup. This potential is constructed as follows. For each $\zeta \in \mathfrak{g}$ and fixed point component $X^{\zeta,t} \subset X^\zeta$ that is t-semistable for some $t \in (-1,1)$, let $\overline{\mathcal{M}}_n^{G_\zeta}(\mathbb{P}, X, L_\pm, \zeta, t, d)$ denote the stack of Mundet semistable morphisms $\mathbb{P} \to X/G_\zeta$ that are $\mathbb{C}_\zeta^\times$-fixed and take values in $X^{\zeta,t}$ on the principal component. Note that the bubbles can take values in the whole of X. Let $\operatorname{Ind}(T(X/G))^+ \subset \operatorname{Ind} T(X/G))$ denote the moving part of the normal complex $\operatorname{Ind}(T(X/G))$ with respect to the action induced by ζ, considered as an object in the derived category of bounded complexes of coherent sheaves on

$$\overline{\mathcal{M}}_n^{G_\zeta}(\mathbb{P}, X, L_\pm, \zeta, t, d).$$

Virtual integration over the stacks $\overline{\mathcal{M}}_n^{G_\zeta}(\mathbb{P}, X, L_\pm, \zeta, t, d)$ defines a "fixed point contribution"

$$\tau^{G_\zeta}_{X,\zeta,t} : QH_G(X) \to \Lambda^G_X[\xi, \xi^{-1}],$$

which sends α into

$$\sum_{d,n\geq 0} \int_{[\overline{\mathcal{M}}_n^{G_\zeta}(\mathbb{P},X,L_\pm,\zeta,t,d)]} \mathrm{ev}^*(\alpha,\ldots,\alpha) \cup \mathrm{Eul}_{\mathbb{C}^\times_\zeta}(\mathrm{Ind}(T(X/G))^+)^{-1} \frac{q^d}{n!},$$

where again $H_{\mathbb{C}^\times_\zeta}(pt) = \mathbb{Q}[\xi]$.

The wall crossing formula (11) can be thought of a generalisation of the wall-crossing formula for abelian vortices studied in [11]. In order to prove a wall crossing formula for the GW invariants of the quotient, we need to take an adiabatic limit in the polarisation.

4.2 Adiabatic limits.

Adiabatic limits appear as special cases of the wall crossing formula (11), when the polarisations are just a rescaling of each other, $L_+ = L_-^t$, $L = L_-$, so that the variation of semi-stability depends only on the rational $t \in (0, \infty)$. In the symplectic category, this is equivalent to the dependence on the "area form", as we originally discussed it in [19].

4.2.1 Small area limit, $t \to 0$.

The moduli stack of Mundet semistable gauged maps in this limit is identified with a quotient of the moduli stack of parametrized stable maps to X. Let $C \cong \mathbb{P}$ be a curve of genus zero. A stable map $u = (u_C, u_X) : \hat{C} \to C \times X$ of degree $(1, d)$, $d \in H_2(X)$ is *zero-semistable* if and only if $u_C(u_X^{-1}(X^{\mathrm{ss}}))$ is dense in C, that is, if it is generically semi-stable. Denote by $\overline{\mathcal{M}}_n(C, X, d)^{0\text{-ss}} \subset \overline{\mathcal{M}}_n(C, X, d) := \overline{\mathcal{M}}_{0,n}(C \times X, (1, d))$ the 0-semistable locus and by

$$\overline{\mathcal{M}}_n(C, X, d) /\!\!/_0 G = \overline{\mathcal{M}}_n(C, X, d)^{0\text{-ss}}/G \tag{12}$$

the quotient stack, of the G action given by post-composition.

Theorem 4.2.1. *[22, Theorem 1.2] With the considerations above, let $d \in H_2(X; \mathbb{Z}) \subset H_2^G(X, \mathbb{Z})$. There exists a t_0 such that for $t < t_0$, there is an isomorphism $\overline{\mathcal{M}}_n(C, X, d) /\!\!/_0 G \to \overline{\mathcal{M}}_n^G(C, X, d; L_t)$ of Deligne-Mumford stacks equipped with perfect relative obstruction theories.*

The proof relies on the following. The rescaling of the polarisation $L \to L^t$ has a linear effect on the Mumford term in the Mundet's weight (7): $\mu_M(\sigma, \lambda) = t\mu_{HM}(\sigma, \lambda) + \mu_R(\sigma, \lambda)$. For t sufficiently small, Mundet semi-stability is thus equivalent to semi-stability of the G-bundle and therefore semi-stability of a gauged map reduces to that of the bundle. Since C is rational this is equivalent for the bundle P to be the trivial.

The relative obstruction theory on $\overline{\mathcal{M}}_n^G(C, X)$ has complex $(Rp_* e^* T(X/G))^\vee$ given by descent from $(\mathfrak{g} \to T\overline{\mathcal{M}}_n(C, X))^\vee$, since every t-semistable map has underlying trivial bundle. The latter is isomorphic to the relative obstruction theory on $\overline{\mathcal{M}}_n(C, X) /\!\!/ G$. For abelian actions on affine spaces, this limit is related to the *toric map spaces* of Givental [16, Section 5]. An analogous formula as in the potential (16), but integrating over the moduli spaces $\overline{\mathcal{M}}_n(\mathbb{P} \times X, (1, d)) /\!\!/ G$ yields a $(t = 0)$ potential

$$\text{(13)} \qquad \textbf{(Quantum Witten Trace)} \qquad \tau_X^G : QH_G(X) \to \Lambda_X^G$$

which is called in [19] the *quantum Witten trace,* since when $d = 0$, it recovers the trace Witten suggested for his localisation (4). Combining with Theorem 4.2.1 we obtain:

$$\lim_{t \to 0} \tau_{X, L_t}^G = \tau_X^G. \tag{14}$$

In this "small area" limit, we can express the gauged potential in terms of morphisms from stable map spaces, at least in the case when C has genus zero. Let

$$\kappa_{\overline{\mathcal{M}}, G} : H^G(\overline{\mathcal{M}}_n(C, X, d) \to H(\overline{\mathcal{M}}_n(C, X, d) /\!\!/ G);$$

$$\tau_{\overline{\mathcal{M}} /\!\!/ G} : H(\overline{\mathcal{M}}_n(C, X, d) /\!\!/ G) \to \mathbb{Q}$$

denote the Kirwan map (that is, restriction to the 0-semistable locus and descent) and virtual integration respectively. Then τ_X^G is the the composition of pull-back with integration over the moduli space of stable maps

$$\tau_X^G : H_G(X) \to \Lambda_X^G, \quad \alpha \mapsto \sum_{d,n} (q^d/n!) \tau_{\overline{\mathcal{M}} /\!\!/ G} \circ \kappa_{\overline{\mathcal{M}}, G} \operatorname{ev}^*(\alpha, \ldots, \alpha).$$

For the purposes of the next section, we remark that if $C \cong \mathbb{P}$ is equipped with a $\mathbb{C}^\times$-action then the same results hold for the $\mathbb{C}^\times$-equivariant potentials.

4.2.2 Large area limit, $t \to \infty$ and the quantisation of the Kirwan map.

This case has been treated by Woodward [45, 46, 47] in the algebraic case, and by Ziltener [49] in the symplectic case, based on earlier work of Gaio-Salamon [18]. In this limit, the bundle contribution of the Mundet weight (7) vanishes, so that gauged maps take values in the semi-stable quotient $X^{\mathrm{ss}}/G = X /\!\!/ G$. One needs to add some other corrections accounting for special affine gauged bubbles which arise in this case, contrasting to the small limit case where such special bubbling in the limit does not appear. More precisely, in this limit the gauged potential (10) factorises [45, Theorem 1.5] as
(15)
(Adiabatic limit Factorisation) $$\lim_{\delta\to\infty} \tau_X^G(\alpha) = \tau_{X/\!\!/G} \circ \kappa_X^G(\alpha),$$

that is, the diagram

$$\begin{tikzcd} QH_G(X) \arrow[rr, "\kappa_X^G"] \arrow[dr, "\tau_X^G"'] & & QH(X /\!\!/ G) \arrow[dl, "\tau_{X/\!\!/G}"] \\ & \Lambda_X^G & \end{tikzcd}$$

commutes in the limit. Here the map

(16) **(Graph Potential)** $\tau_{X/\!\!/G} : QH(X /\!\!/ G) \to \Lambda_X^G$;

$$\tau_{X/\!\!/G}(\alpha) := \sum_{n\geq 0} \sum_{d\in H_2(X/\!\!/G;\mathbb{Q})} \left(\int_{\overline{\mathcal{M}}_n(\mathbb{P},X/\!\!/G,d)} \mathrm{ev}^*(\alpha,\dots,\alpha) \right) \frac{q^d}{n!}$$

is the *graph potential* that counts *parametrised* maps into $X /\!\!/ G$ in the class d and the special bubbling in the limit is encoded by the *quantum Kirwan map* κ_X^G. This map is defined by virtual enumeration of *affine gauged maps*: pairs (u, λ) where u is a representable morphisms $u : \mathbb{P}(1, r) \to X/G$ from a weighted projective line $\mathbb{P}(1, r), r > 0$ to the quotient stack X/G, mapping the stacky point at infinity $\mathbb{P}(r) \subset \mathbb{P}(1, r)$ to the semistable locus $X /\!\!/ G \subset X/G$. These are the algebro-geometric analogues of the *vortex bubbles* considered in Gaio-Salamon [18]. λ is a *scaling*, a meromorphic 2-form $\lambda : T^\vee\mathbb{P} \to \mathcal{O}(2\infty)$ with double pole at ∞. The compactified moduli stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X, d)$ of affine gauged maps of homology class $d \in H_2^G(X, \mathbb{Q})$ is, if stable=semi-stable for the G action on X, a proper smooth Deligne-Mumford stack with a perfect relative obstruction theory over the complexification of Stasheff's multiplihedron: the moduli $\overline{\mathcal{M}}_{n,1}(\mathbb{A})$ of n-marked scaled lines (with a single scaling). It is equipped with evaluation maps $\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}^G_{n,1}(\mathbb{A}, X) \to (X/G)^n \times (X /\!\!/ G)$. The quantum Kirwan map is the formal morphism

(17) **(Quantum Kirwan map)** $\kappa^G_X : QH_G(X) \to QH(X /\!\!/ G)$;

$$\kappa^G_X(\alpha) := \sum_{n \geq 0, d} \mathrm{ev}_{\infty,*}\, \mathrm{ev}^*(\alpha, \ldots, \alpha) \frac{q^d}{n!}.$$

It is in principle non-linear, and it is a morphism of CohFT theories as detailed in [45, Section 2.3], [47, Theorem 8.6] and Woodward-Ziltener [48]. Moreover, each of its linearisations $D_\alpha \kappa^G_X : T_\alpha QH_G(X) \to T_{\kappa(\alpha)} QH(X /\!\!/ G)$ is a homomorphism with respect to the quantum product. We are now ready to apply the wall crossing formula in several contexts.

Example 4.2.2. Let us analyse the toric case. Let $X = \mathbb{C}^N$ be the dimensional complex vector space with an action of a complex torus $T = (\mathbb{C}^\times)^k$, with weights $\mu_1, \ldots, \mu_N$ contained in an open half-space and equipped with a polarisation so that the quotient $X /\!\!/ T$ is a Deligne-Mumford stack. First we analyse the affine gauged maps appearing in the definition of κ^T_X: An affine gauged map to X/T of homology class $d \in H^T_2(X, \mathbb{Q})$ is equivalent to a morphism $u = (u_1, \ldots, u_k) : \mathbb{A} \to X$ such that the degree of u_j is at most $d_j = \langle \mu_j, d \rangle$. Let $u_j(\infty) = u_j^{d_j}/(d_j!)$ if $d_j \geq 0$ and $u_j(\infty) = 0$ otherwise. Then, let $u(\infty) = (u_j(\infty))^k_{j=1}$ denote the vector of leading order coefficients with integer exponents, then $u(\infty) \in X^{\mathrm{ss}}$. Thus $\overline{\mathcal{M}}^T_{1,1}(\mathbb{A}, X, d)$ is the space of such morphisms with one finite marking and the marking at infinity up to the action of T. We let $\overline{\mathcal{M}}^T_{1,0}(\mathbb{A}, X, d)$ denote the space of morphisms with only the marking at infinity up to automorphisms. In this case the automorphisms do not only come from the action of T, but also from the translation of the domain, since there is no finite marking. If $T = \mathbb{C}^\times$ acts on $X = \mathbb{C}^N$ by scalar multiplication then $\overline{\mathcal{M}}^T_{1,1}(\mathbb{A}, X, d)$ consists of tuples of polynomials of degree at most d up to the action of T, such that at least one of the polynomials is of degree exactly d. One sees that $\overline{\mathcal{M}}^T_{1,1}(\mathbb{A}, X, d)$ is a vector bundle over $\mathbb{P}^{N-1}$ of rank dN.

Now, let $G = (\mathbb{C}^\times)^N$ be the big torus acting in the standard way in X and consider the equivariant cohomologies $QH_T(X) \cong \mathrm{Sym}(\mathfrak{t}^\vee) \otimes$

Λ_X^T, $QH_G(X) \cong \mathrm{Sym}(\mathfrak{g}^\vee) \otimes \Lambda_X^G$. By taking coordinates $v_1, \dots v_N$ on $\mathfrak{g}$ we get that $QH_G(X) = \mathbb{Q}[v_1, \dots, v_N] \otimes \Lambda_X^G$. Let $p : QH_G(X) \to QH_T(X)$ be the restriction map and let $D(\mu_i) \in H^2(X /\!\!/ T)$ denote the divisor classes defined by v_i. Then, the degree one portion $\kappa_X^{G,1}$ of the Kirwan map is given by [45, Lemma 8.8] (see [20] for the stacky case)

$$\kappa_X^{T,1}\left(\prod_{\mu_j(d)\geq 0} p(v_j)^{\mu_j(d)}\right) = q^d \prod_{\mu_j(d)\leq 0} D(v_j)^{-\mu_j(d)} + \text{ higher order terms,}$$

where $d \in H_2^T(X, \mathbb{Z})$ is a lift of an effective class. This yields the Batyrev quantum relations for the quotient $X /\!\!/ T$. The surjectivity of the quantum Kirwan map for toric stacks follows similarly. The reader can consult [20] for more details, as well as the computation of the quantum cohomology of toric stacks with projective coarse moduli space.

4.2.3 Relation to the I and J functions

We continue with the same notations $T = (\mathbb{C}^\times)^k$, acting on $\mathbb{C}^N$. Woodward's adiabatic limit (15) can be understood as a generalisation of Givental's mirror symmetry result [16]. In this case the gauged potential plays the rôle of the I function and the graph potential of the J function, whit the quantum Kirwan map as the mirror transformation. To obtain the actual I, J functions, we need to localise the potentials, as follows. Consider the case $C = \mathbb{P}$ equipped with the standard $\mathbb{C}^\times$-action given by rotation, with fixed points $0, \infty \in \mathbb{P}$. Let $\hbar$ denote the equivariant parameter. The graph potential $\tau_{X/\!\!/T}$ extends to a $\mathbb{C}^\times$-equivariant potential $\tau_{X/\!\!/T,\mathbb{C}^\times} : QH(X /\!\!/ G) \to \Lambda_X[\![\hbar]\!]$. The *localised graph potential* $\tau_{X/\!\!/T,-} : QH(X /\!\!/ T) \to QH(X /\!\!/ T)[\![\hbar^{-1}]\!]$ is the localisation to the fixed point $0 \in \mathbb{P}$. $\tau_{X/\!\!/T,-}$ is a solution to the fundamental quantum differential equation $\hbar\partial_\upsilon \tau_{X/\!\!/T,-}(\alpha) = \upsilon *_\alpha \tau_{X/\!\!/T,-}(\alpha)$ for the Frobenius manifold associated to the GW theory of $X /\!\!/ T$. Localising the gauged potential $\tau_{X,-}^T$ as well, and by considering a $\mathbb{C}^\times$-equivariant extension of the quantum Kirwan map κ_X^T, we have the *localised adiabatic limit* theorem

$$\tau_{X/\!\!/T,-} \circ \kappa_X^T = \lim_{t\to\infty} \kappa_X^{T,\mathrm{class}} \circ \tau_{X,-}^T, \tag{18}$$

where $\kappa_X^{T,\mathrm{class}}$ is the classical Kirwan map. In the toric case $X = \mathbb{C}^N, T = (\mathbb{C}^\times)^k$, the right term agrees with Givental's I function,

$$\kappa_X^{T,\mathrm{class}} \circ \tau_{X,-}^T(\alpha) = \exp(\alpha/\hbar) \sum_{d \in H_2^T(X)} q^d \prod_{j=1}^N \frac{\prod_{m=-\infty}^0 (\mu_j + m\hbar)}{\prod_{m=-\infty}^{\mu_j(d)} (\mu_j + m\hbar)}, \tag{19}$$

and the localised graph potential $\tau_{X/\!\!/T,-}$ is the J function, by Givental [15]. This implies that the quantum Kirwan map plays the rôle of the mirror map. For more details see [45, Section 8].

4.3 Wall-crossing for GW invariants of quotients.

We will follow the same notations we had in Section 2.1. We now explain how the quantised version of Kalkman's wall crossing Equation (3), holds in quantum cohomology. Our wall crossing formula for Gauge GW invariants and the quantum Kirwan map descends to one for GW invariants of $X /\!\!/ G$. We may think of gauged maps as interpolating between maps to the quotient and quotient of maps, thus taking both of the polarisations to the adiabatic large limit, and applying the factorisation of Equation (15) we obtain.

Theorem 4.3.1 (Theorem 3.5 [21])**.** *Applying the wall-crossing formula for two polarisations L_+^t and L_-^t and taking the limit $t \to \infty$ we have*

(20) ***(Wall-Crossing for GW potentials)*** $\tau_{X/\!\!/_+G} \circ \kappa_{X,+}^G(\alpha) -$

$$\tau_{X/\!\!/_-G} \circ \kappa_{X,-}^G(\alpha) = \sum_\zeta \operatorname{Res}\ \tau_{X,\zeta,L^\delta}(\alpha).$$

In general the terms on the right of Equation (20) are gauged Gromov-Witten invariants, which in principle can be expressed in terms of the usual Gromov-Witten invariants of the smaller quotients on the walls, just as in classical Kalkman's formula. However gauged invariants are easier to compute, so we will leave the formula as is, since it is easier to understand the effect of crossing a wall for the crepant case to be discussed in Section 4.4.

Remark 4.3.2. The formulas just presented are in their most basic form. In general one can add insertions from classes $\beta \in H(\overline{\mathcal{M}}_n(C))$ as well as a twisting by Euler classes of index bundles. This is in particular important for explicit computations of particular invariants.

We end this discussion with an example.

Example 4.3.3. We use the same notation as in Example 3.1.6. Let $G = \mathbb{C}^\times$ acting on $X = \mathbb{C}^N$ by scalar multiplication, $H_G(X) = \mathbb{Q}[\xi]$. And $L_\pm$ are the lines with weights ± 1, inducing $X /\!\!/_- G = \emptyset$ and $X /\!\!/_+ G = \mathbb{P}^{N-1}$. The wall is $t = 0$ corresponding to the point $0 \in X$. We compute genus zero, degree one three-point invariants $\left\langle h^a, h^b, h^c \right\rangle_{0,1}$ of $\mathbb{P}^{k-1}$ via wall-crossing. Since the equivariant chern class $c_1^G(X) = N\xi$, the minimal Chern number of X is N, and this implies that the Kirwan map

$$D_0 \kappa_X^G(\xi^i) = h^i, i \leq N-1.$$

By dimension reasons, there are no quantum Kirwan corrections, and thus the quantum Kirwan map agrees with the classical one. The adiabatic limit (15) with insertions from the Fulton-McPherson space implies that the three-point invariants in the quotient equal the gauged Gromov-Witten invariant,

$$\left\langle h^a, h^b, h^c \right\rangle_{0,1} = \int_{[\overline{\mathcal{M}}_3^G(\mathbb{P},X,d)]} \mathrm{ev}_1^* \xi^a \cup \mathrm{ev}_2^* \xi^b \cup \mathrm{ev}_3^* \xi^c \cup f^* \beta$$

where $\beta \in H(\overline{\mathcal{M}}_3(\mathbb{P}))$ is the class fixing the location of the three marked points. Consider the wall-crossing Formula (20). There are no holomorphic spheres in X, so the moduli stack $\overline{\mathcal{M}}_0^{G_\zeta}(\mathbb{P}, X, L_\pm, \zeta, t, d)$ is a point, consisting of the bundle P with first Chern class $c_1(P) = d \in H_G^2(X, \mathbb{Z}) \cong \mathbb{Z}$ $(d = 1)$ with constant section equal to zero. Thus $\overline{\mathcal{M}}_3^{G_\zeta}(\mathbb{P}, X, L_\pm, \zeta, t, d) \cong \overline{\mathcal{M}}_3(\mathbb{P})$. By wall-crossing

$$\begin{aligned}
\left\langle h^a, h^b, h^c \right\rangle_{0,1} &= \mathrm{Res}_\xi \int_{[\overline{\mathcal{M}}_3(\mathbb{P},L_-,L_+,X,\zeta,t,1)]} \frac{\mathrm{ev}_1^* \xi^a \cup \mathrm{ev}_2^* \xi^b \cup \mathrm{ev}_3^* \xi^c \cup f^* \beta}{\mathrm{Eul}(\mathrm{Ind}(T(X/G))^+)} \\
&= \mathrm{Res}_\xi \, \xi^{a+b+c}/\xi^{2k} \\
&= \begin{cases} 1 & a+b+c = 2N-1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

which counts the unique line passing through two generic points and a generic hyperplane.

4.4 A GIT version of the crepant conjecture.

We now explain how we can use the wall-crossing formula to prove a special case of the crepant conjecture. A birrational transformation of GIT type induced by a wall-crossing L_+, L_-

$$\phi : X /\!\!/_- G \dashrightarrow X /\!\!/_+ G$$

will be called *crepant* if $c_1^{\mathbb{C}^\times}(NF) = 0$ for each fixed component $F \subset M^{\mathbb{C}^\times}$ of the master space, and *weakly crepant* if the sum of the weights of $\mathbb{C}^\times$ on NF, counted with multiplicity, vanishes. Recall that the fixed point sets of the master space are in correspondence with fixed sets X^ζ by one-parameter subgroups as discussed in Section 2.1.

The contribution from a fixed point component $X^{\zeta,t}$ to the Formula (20) is given by the fixed point potential $\tau_{X,\zeta,t} : QH_G(X) \to \Lambda_G[\xi, \xi^{-1}]$, which in turn is given by twisted gauged invariants coming from the fixed moduli $\overline{\mathcal{M}}_n(\zeta, t, d) := \overline{\mathcal{M}}_n^{G_\zeta}(C, X, L_\pm, \zeta, t, d)$, where we reduce the notation for simplicity. $\overline{\mathcal{M}}_n(\zeta, t, d)$ consists of tuples $(P, \hat{C}, u)$ where $P \to C$ is a G-bundle and $u : \hat{C} \to P(X)$ is ζ-fixed, in particular, the restriction of u to the principal component of C maps into the locus X^ζ. Let $\mathbb{C}_\zeta^\times \subset G_\zeta$ denote the subgroup of G_ζ generated by $\zeta \in \mathfrak{g}_\zeta$ and identify $\mathrm{Pic}(C) = \mathrm{Maps}(C, B\mathbb{C}_\zeta^\times) \cong \mathbb{Z}$ with the group of isomorphism classes of $\mathbb{C}_\zeta^\times$-bundles $P_\zeta \to C$. There is a canonical action of the Picard group $\mathrm{Pic}(C)$ on the moduli stack $\overline{\mathcal{M}}_n(\zeta, t, d)$, given by

$$P_\zeta(P, \hat{C}, u) = (P \times_{\mathbb{C}_\zeta^\times} P_\zeta, \hat{C}, u)$$

where we use that

$$(P \times_{\mathbb{C}_\zeta^\times} P_\zeta)(X^\zeta) \cong (P \otimes \mathcal{O}_C(d))(X^\zeta)$$

since the action of $\mathbb{C}_\zeta^\times$ on X^ζ is trivial. When restricting to the part induces an isomorphism shifting degree

$$\mathcal{S}_\delta : \overline{\mathcal{M}}_n(\zeta, t, d) \to \overline{\mathcal{M}}_n(\zeta, t, d + \delta) \tag{21}$$

where $\delta = c_1(P_\zeta)$ is the generator. The action of $\mathrm{Pic}(C)$ lifts to the universal curves, and, since the obstruction theory on $\overline{\mathcal{M}}_n(\zeta, t, d)$ is the ζ-invariant part of $\mathrm{Ind}(T(X/G))$, the isomorphism preserves the relative obstruction theories and so the Behrend-Fantechi virtual fundamental classes. Since the evaluation map is unchanged, the class $\mathrm{ev}^* \alpha$ is preserved for any $\alpha \in H_{G_\zeta}(X)^n$.

The action of $\mathrm{Pic}(C)$ helps us understand the fixed point contributions $\tau_{X,\zeta,d,t}$. The contribution of any component $\overline{\mathcal{M}}_n(\zeta, t, d)$ of class $d \in H_2^G(X)$ differs from that from the component induced by acting by P_ζ, of class $d + \delta$, by the difference in Euler classes $\mathrm{Eul}(Rp_* \, \mathrm{ev}^* \, T(X/G)^+)$ and $\mathrm{Eul}(S_\delta^* Rp_* \, \mathrm{ev}^* \, T(X/G)^+)$. We now compute this difference. Assume for simplicity that the component $X^{\zeta,t}$ of the fixed point set X^ζ

which is semistable for t is connected (repeat the following argument for each connected component). Consider the decomposition into $\mathbb{C}^\times_\zeta$-bundles

$$\nu_{X^{\zeta,t}} = \bigoplus_i^k \nu_{X_i^{\zeta,t}}$$

where $k = \operatorname{codim}(X^{\zeta,t})$ and $\mathbb{C}^\times_\zeta$ acts on $\nu_{X_i^{\zeta,t}}$ with non-zero weight $\mu_i \in \mathbb{Z}$. Then $\operatorname{ev}^* T(X/G_\zeta)$ is isomorphic to $\mathcal{S}_\delta \operatorname{ev}^* T(X/G)$ on the bubble components, since the G_ζ-bundles are trivial, while on the principal component

$$(e^*T(X/G))^+ \cong \bigoplus_i e^*\nu_{X_i^{\zeta,t}} \text{ and}$$

$$S^*_\delta(e^*T(X/G))^+ \cong \bigoplus_i e^*\nu_{X_i^{\zeta,t}} \otimes \operatorname{ev}_C^* \mathcal{O}_C(\mu_i\delta)$$

where $\mathcal{O}_C(1)$ is a hyperplane bundle on C. Hence the difference between the pull-back complexes vanishes on the bubble components and so pushes forward to a complex on the principal part of the universal curve

$$p_0 : C \times \overline{\mathcal{M}}_n(\zeta,t,d) \to \overline{\mathcal{M}}_n(\zeta,t,d).$$

This is a representable morphism of stacks, which admit a presentation as global quotients embedded in smooth DM stacks [1, Theorem 1.0.2] in which case by Grothendieck-Riemann-Roch [14, Theorem 3.1] ,

$$\mathcal{S}^*_\delta \operatorname{Ind}(T(X/G))^+ \cong \operatorname{Ind}(T(X/G))^+ \oplus \bigoplus_i (z^*e^*\nu_{X_i^{\zeta,t}})^{\oplus \mu_i\delta} \tag{22}$$

where $z : \overline{\mathcal{M}}_n(\zeta,t,d) \to C \times \overline{\mathcal{M}}_n(\zeta,t,d)$ is a constant section of p_0. By the splitting principle we may assume that the $\nu_{X_i^{\zeta,t}}$ are line bundles. The difference in Euler classes

$$\operatorname{Eul}(\mathcal{S}^*_\delta \operatorname{Ind}(T(X/G))^+)\operatorname{Eul}((\operatorname{Ind}(T(X/G))^+)^{-1} \in H_{\mathbb{C}^\times}(\overline{\mathcal{M}}_n(\zeta,t,d))$$

is given by the Euler class of the last summand in (22)

$$\operatorname{Eul}\left(\bigoplus_{i=1}^k (z^*e^*\nu_{X_i^{\zeta,t}})^{\oplus\mu_i\delta}\right) = \prod_{i=1}^k (\mu_i\xi + c_1(\nu_{X_i^{\zeta,t}}))^{\mu_i\delta} =$$

$$\prod_{i=1}^k (\xi + c_1(\nu_{X_i^{\zeta,t}})/\mu_i)^{\mu_i\delta} \prod_{i=1}^k \mu_i^{\mu_i\delta}$$

factoring out the weights μ_i. Let $\mu = \sum_{i=1}^k \mu_i$. Expanding out the product we obtain

$$\prod_{i=1}^k (\xi + c_1(\nu_{X_i^{\zeta,t}})/\mu_i)^{\mu_i \delta} = \xi^{\delta\mu} + \xi^{\delta\mu-1}\left(\delta \sum_{i=1}^k c_1(\nu_{X_i^{\zeta,t}})\right) + \tag{23}$$

$$\xi^{\delta\mu-2}\left(\delta^2 \sum_{i\neq j} c_1(\nu_{X_i^{\zeta,t}}) c_1(\nu_{X_j^{\zeta,t}}) + \sum_{i=1}^k \delta(\delta - 1/\mu_i) c_1(\nu_{X_i^{\zeta,t}})^2\right) + \ldots$$

and $\ldots$ indicates further terms that are polynomials in $\xi, \xi^{-1}, \delta, \mu_i$ and μ_i^{-1}. If the wall crossing is crepant, the sum of the weights in the normal directions vanish, so $\mu = \sum_{i=1}^k \mu_i = 0$, and in (23), the exponents $\delta\mu$ vanish. Adding all expressions (23) over δ, we obtain the wall crossing term

$$\sum_{\delta\in\mathbb{Z}} \operatorname{Resid}_\xi q^{d+\delta} \tau_{X,\zeta,d+\delta,t} = \tag{24}$$

$$\sum_{\delta\in\mathbb{Z}, t\in(-1,1)} \prod_{i=1}^k \mu_i^{\mu_i\delta} q^{d+\delta} \int_{[\overline{\mathcal{M}}_n(\zeta,d,t)]} (1 + \xi^{-1} \sum_{i=1}^k \delta c_1(\nu_{X_i^{\zeta,t}}) + \ldots) \operatorname{ev}^* \alpha.$$

Now for any integers $m_i \geq 0, n_i, i = 1, \ldots, k$

$$\sum_\delta \prod_{i=1}^k \delta^{m_i} \mu_i^{\mu_i \delta - n_i} q^{d+\delta} =_{a.e.} 0$$

vanishes almost everywhere in q, being a function times a sum of derivatives of delta functions in q. (Here recall that $\sum_{n\in\mathbb{Z}} q^n = \delta(q = 1)$, the delta distribution supported at $q = 1$.) Almost everywhere equality in a formal parameter q means the following: after consider both sides as elements in $\operatorname{Hom}(H_2^G(X,\mathbb{Z})/\operatorname{torsion}, \mathbb{Q})$ the space of distributions on $H_G^2(X,\mathbb{R})/H_G^2(X,\mathbb{Z})$, the difference is tempered in at least one direction and its Fourier transform in that direction has support of measure zero. Since $\tau_{X/\!\!/_+G} - \tau_{X/\!\!/_-G}$ is a sum of such wall-crossing terms, the discussion above justifies the following result.

Theorem 4.4.1 (Theorem 1.13 [21])**.** *If the all the wall-crossings are weakly crepant then*

$$\tau_{X/\!\!/_-G} \circ \kappa_{X,-}^G =_{a.e} \tau_{X/\!\!/_+G} \circ \kappa_{X,+}^G$$

almost everywhere (a.e.) in the formal parameters q.

Note that this result is *not* a quantisation of any classic result in equivariant cohomology and it is purely a quantum cohomology result. Theorem 4.4.1 is *a version* of the crepant resolution conjectures of Li-Ruan [32], Bryan-Graber [6], Coates-Ruan [10]. It is important to remark that for derived categories (B-model) the work [24, 25], brings equivalences for derived categories of sheaves in the GIT case where the same crepant condition of weights is found.

4.5 Quantum Witten localisation and quantum abelianisation

We now discuss a quantum version of the abelianisation formula relating traces of the quotients $X /\!\!/ G, X /\!\!/ T$, as discussed in section 2.2. We first need to introduce a quantum version of Witten's non-abelian localisation.

Theorem 4.5.1 (Theorem 1.0.2 [19]). *Under suitable stable=semistable conditions, the following formula holds*

(25)

(Quantum Witten localisation) $$\tau_X^G = \tau_{X/\!\!/G} \circ \kappa_X^G + \sum_{[\zeta], t\in(0,\infty)} \tau_{X,G,\zeta,L^t}.$$

Theorem 4.5.1 follows from the wall crossing formula (11) and by taking the large (15), and small (14) limits simultaneously. The contributions on the right correspond to contributions described in (11) from all components fixed by one-parameter subgroups for $t \in (0,\infty)$. Here we are assuming stable=semistable with respect to the smaller groups on each of the fixed moduli that contributes. As mention above, our original intention for Equation (25) was to show abelianisation for quantum cohomology. Recall from Section 2.2 the notations and discussion on Martin's formula (5). Let $T \subset G$ be a maximal torus, and suppose stable=semistable for the actions of T, G on the projective variety X. Let W denote the Weyl of T and let $\mathrm{Restr}_T^G : H_G(X) \to H_T(X)$, be the natural restriction. Then, integration over $X /\!\!/ G$ and $X /\!\!/ T$ are related by

$$\tau_{X/\!\!/G} \circ \kappa_X^G = |W|^{-1} \tau_{X/\!\!/T} \circ \kappa_X^T \circ \mathrm{Restr}_T^G .$$

The abelianisation result states that in quantum cohomology a similar equation holds, after we change the traces by quantised potentials.

Theorem 4.5.2 (Theorem 1.0.3 [19]). *Under suitable stable=semistable* *conditions,*

$$\tau_{X/\!\!/G} \circ \kappa_X^G = |W|^{-1} \pi_T^G \circ \tau_{X/\!\!/T} \circ \kappa_X^T \circ \mathrm{Restr}_T^G \,. \tag{26}$$

Recall that the potential $\tau_{X/\!\!/T}$ is twisted by the Euler class of the index bundle $\mathrm{Index}(\mathfrak{g}/\mathfrak{t})$ induced by the roots, the κ's denote quantum Kirwan maps. The push-forward $\pi_T^G : H_2^T(X) \to H_2^G(X)$ induces one of equivariant Novikov variables $\Lambda_X^T \to \Lambda_X^G$ (see (9)) by

$$\pi_T^G : \Lambda_X^T \to \Lambda_X^G; \quad \sum_{d \in H_2^T(X)} c_d q^d \mapsto \sum_{d \in H_2^T(X)} c_d q^{\pi_T^G(d)}.$$

The map $\mathrm{Restr}_T^G : H_G(X) \to H_T(X)$ induces the isomorphism $H_G(X) \cong H_T(X)^W$ and together with the map π_T^G in Novikov variables we get, the map $QH_G(X) \to QH_T(X)$, by extending to quantum cohomology.

The proof of the theorem is iterative. First we prove it in the small limit [22, Theorem 1.3], that is

$$\tau_X^G = |W|^{-1} \pi_T^G \circ \tau_X^T.$$

In the small chamber, abelianisation is somehow easier and similar in nature to the original proof given by Martin (Section 2.2) since the moduli spaces are essentially "GIT quotients" of stacks in the sense of Equation (12). There is a technical detail in this proof, namely, we needed to restrict to $QH_G^{\mathrm{chow}}(X) \subset QH_G(X)$, the portion of classes which are algebraic. This is because we rely on Grothendieck-Riemann-Roch for sheaf cohomology on stacks. (Although we believe that this restriction is not necessary, see [22, Section 5] for more details.) Once the proof of abelianisation in the small limit is established, the proof of Theorem follows by iteration, increasing the scaling t to pass to the big limit while crossing all walls in between. We then check that the contributions for both quotients arising on the walls agree. To be more precise, we carefully compare the wall terms in the abelian and non-abelian quantum Witten localisation formulas

$$\tau_X^G - \tau_{X/\!\!/G} \circ \kappa_{X,G} = \sum_{[\zeta] \neq 0,t} \tau_{X,G,\zeta,t} \quad \text{and} \quad \tau_X^T - \tau_{X/\!\!/T} \circ \kappa_{X,T} = \sum_{[\zeta] \neq 0,t} \tau_{X,T,\zeta,t}$$

Again, we remark that all the abelian potentials are twisted by the Euler class of the index bundle $\mathrm{Ind}(\nu_{\mathfrak{g}/\mathfrak{t}})$. This now gives the abelianisation formula in the large limit, that is, for the GIT quotients.

Example 4.5.3. We consider the Grassmann variety $\mathrm{Gr}(k,n)$ of k-planes in the affine space $\mathbb{C}^n$ of dimension n. We present $\mathrm{Gr}(k,n)$ in the standard GIT manner as follows. Let GL_k act on $X = \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ by $(g \cdot x)(v) = x(g^{-1}v)$ for all $x \in X, v \in \mathbb{C}^k$ and $g \in G$. Since $H^2_G(X,\mathbb{Q}) \cong \mathbb{Q}$, we take the linearisation corresponding to $c_1^G(X)$. The semistable locus X^{ss} is given by the injective maps (full rank matrices) in X. Then $\mathrm{Gr}(k,N) = X /\!\!/ GL_k$.

Consider the maximal torus $T = (\mathbb{C}^\times)^k$ of invertible diagonal matrices in G. The Weyl group $W = S_n$ acts on the equivariant cohomology $H_T(X;\mathbb{Q}) = \mathbb{Q}[\theta_1, \dots, \theta_k]$ and the invariant part $H_T(X;\mathbb{Q})^W$ is identified with $H_G(X;\mathbb{Q})$. The roots are then identified with $\theta_i - \theta_j$ for $i < j$, and the first Chern class $c_1^G(X) = n(\theta_1 + \cdots + \theta_k)$, thus it is positive, meaning that the equivariant Chern class takes postive values on all degrees for which there is a gauged map. Positivity (monotonity) of X implies that there are no quantum corrections coming from $H^2_G(X;\mathbb{Q})$ in κ_X^G (see Remark 8.7(b) in [45], or Remark 4.1.1 in [19]). Let t_i denote the coordinates in $H_T(X;\mathbb{Q})$, then by the localised *gauged* potential associated to the torus (19) is

$$\tau_{X,T,-}(t_0 + t_1\theta_1 + \cdots + t_k\theta_k) =$$

$$\exp(t_0 + t_1\theta_1 + \cdots + t_k\theta_k) \sum_{d \in H_2^T(X;\mathbb{Q})} \exp(\sum_i t_i d_i) \tau_{X,T,-}(d) q^d,$$

where the degree $d = d_1\theta_1 + \cdots + d_k\theta_k$ component is given by

$$\tau_{X,T,-}(d) = \prod_{i \neq j, i,j \leq k} \frac{\prod_{m=-\infty}^{d_i - d_j}((\theta_i - \theta_j) + m\hbar)}{\prod_{m=-\infty}^{0}((\theta_i - \theta_j) + m\hbar)} \prod_{j=1}^{k} \frac{\prod_{m=-\infty}^{0}(\theta_j + m\hbar)^n}{\prod_{m=-\infty}^{d_j}(\theta_j + m\hbar)^n},$$

whose first term is the contributions from the roots and the second is n-times repeated. By the abelianisation formula (18), the localised graph potentials on $H^2(\mathrm{Gr}(k,n))$ is given by

$$\tau_{X/\!\!/G,-} = \pi_T^G \circ \kappa_{X,T}^{\mathrm{class}} \circ \tau_{X,-}^T \circ \mathrm{Restr}_T^G, \tag{27}$$

since the quantum Kirwan map has no quantum correction. This is the original formula conjectured by Hori-Vafa [26, Appendix A] and proved by Bertram et al [3, Theorem 1.5] which led to the general Abelian-non-Abelian correspondence [4, Conjectures 1.1, 1.2].

Example 4.5.4. We extend the example above to show that similar computations hold even if the case when the Kirwan map has corrections. In this case however, the corrections can be hard to compute, as expected since they are related to the mirror map. Let V denote a left GL_k-module. We denote by $V_{\mathrm{Gr}(k,n)} \to \mathrm{Gr}(k,n)$ the associated vector bundle $(X^{\mathrm{ss}} \times V)/GL_k$, where the action is diagonal. Let

$$0 \to U \to \mathcal{O}_{\mathrm{Gr}(k,n)} \otimes \mathbb{C}^N \to Q \to 0$$

denote the tautological sequence associated to the universal k-plane bundle $U \to \mathrm{Gr}(k,n)$, where Q is the universal quotient. If W is the standard representation of GL_k, then $W_{\mathrm{Gr}(k,n)} = U$. For our example, consider $l \leq k$, and the projectivisation $\mathcal{F} = \mathbb{P}(\Lambda^l U) \to \mathrm{Gr}(k,n)$ which has a presentation as a GIT quotient

$$(X \times \Lambda^l W) /\!\!/ (GL_k \times \mathbb{C}^\times)$$

where $\mathbb{C}^\times$ acts on the fibres diagonally. The same computation above carries ad-verbatim by noticing that there is an extra generator

$$H_T(X \times \Lambda^l W) = \mathbb{Q}[\theta_1, \ldots, \theta_k, \theta_{k+1}],$$

and that the group $GL_k \times \mathbb{C}^\times$ has the same roots as in Example 4.5.3. Notice that $T \times \mathbb{C}^\times$ acts on W with weights $-\theta_i^\vee, i = 1, \ldots, k$ so on the $\binom{k}{l}$ dimensional space $\Lambda^l W$ with sums of l distinct $-\theta_i^\vee$. Then the associated abelian localised potential twisted by the root contributions is given by essentially the same formula above, by adding t_{k+1}, θ_{k+1} terms appropriately and for $d = d_1\theta_1 + \cdots + d_k\theta_k + d_{k+1}\theta_{k+1}$,

$$\tau_{X \times \Lambda^l W, T \times \mathbb{C}^\times, -}(d) = \prod_{i \neq j, i,j \leq k} \frac{\prod_{m=-\infty}^{d_i - d_j} ((\theta_i - \theta_j) + m\hbar)}{\prod_{m=-\infty}^{0} ((\theta_i - \theta_j) + m\hbar)} \prod_{j=1}^{k} \frac{\prod_{m=-\infty}^{0} (\theta_j + m\hbar)^n}{\prod_{m=-\infty}^{d_j} (\theta_j + m\hbar)^n} \frac{\prod_{m=-\infty}^{0} (\theta_{k+1} + m\hbar)^{\binom{k}{l}}}{\prod_{m=-\infty}^{D_{k+1}(d)} (\theta_{k+1} + m\hbar)^{\binom{k}{l}}}, \tag{28}$$

where $D_{k+1}(d)$ is $d_{k+1} - d_1 - \cdots - d_k$. The fundamental solution is given by the localised graph potential (18) and (26). Here there might be corrections in the Kirwan map, for instance in the semi-positive case, when $n = \binom{k}{l}$. As a more concrete example, consider the Grassmanian

$G(4,6)$ and take $\mathcal{F} = \mathbb{P}(\Lambda^2 U^\vee)$. The maximal torus $T = (\mathbb{C}^\times)^4 \times \mathbb{C}^\times$ acts on $X \times \Lambda^2 W$ with weight matrix

$$A = \begin{pmatrix} \mathbb{1} & 0 & 0 & 0 & -\mathbb{1} \\ 0 & \mathbb{1} & 0 & 0 & -\mathbb{1} \\ 0 & 0 & \mathbb{1} & 0 & -\mathbb{1} \\ 0 & 0 & 0 & \mathbb{1} & -\mathbb{1} \\ 0 & 0 & 0 & 0 & \mathbb{1} \end{pmatrix}$$

Here the $\mathbb{1}$ represents the vector $\mathbb{1} = (1\ 1\ 1\ 1\ 1\ 1)$. Recall that $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in (\mathbb{C}^\times)^4$ acts on $(4,6)$ matrices by multiplying the i-th column by λ_i, with weight 1. Thus $\mathbb{1}$ represent the weights of the action on the entries in each column. $(\mathbb{C}^\times)^4$ acts on the six dimensional space $\Lambda^2 W$ with -1 on each term, these are the $-\mathbb{1}$ in the last column. The last row represents the weights of $\mathbb{C}^\times$ acting on the fibres. Since the space is semi-positive, we need to account the corrections in the Kirwan (mirror) map, and thus the fundamental solution is computed by (26).

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