Morfismos, Vol. 21, No. 2, 2017, pp. 1–16

Bilinear maps, embeddings, topological complexity and antisymmetric index of projective spaces. *

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Abstract

We provide a straight new proof using direct calculations on integral cohomology which can be considered as a substitute, in an infinite number of cases, for a fact concerning bilinear maps, (embeddings) immersions and (symmetric) topological complexity of projective spaces, and equivariant maps.

2010 Mathematics Subject Classification: 57R40, 57R42, 57R19, 55R80, 55M30, 11D09.
Keywords and phrases:Embedding, Immersion, Projective Spaces, Topological Complexity, Equivariant Maps.

1 Introduction

When the classical embedding problem for smooth manifolds is restricted to the case of real projective spaces, it may be seen as a connection between real bilinear, embeddings, and equivariant type problems. In this article it is intended to describe this question starting from some results in the work of A. Haefliger [16] and H. Hopf [18] on the matter.

The existence of a symmetric nonsingular bilinear map $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ gives an embedding of the projective space $\mathbb{R}P^{m-1} \to \mathbb{R}^{k-1}$, and an embedding of a topological space $X \to \mathbb{R}^k$ defines a $\mathbb{Z}_2-$equivariant map $X \times X - \Delta \to S^{k-1}$. In these three cases the main question is,

*This research is supported by DGAPA-UNAM postdoctoral scholarship.

[1]This article contains part of the results of the author's PhD. Thesis directed by Professor Jesús González at Department of Mathematics CINVESTAV-IPN.

what is the minimum k such that there exist these kind of maps? For the equivariant case we refer to this number as the antisymmetric index of X.

If X is a compact manifold, calculating its antisymmetric index and its embedding dimension are equivalent matters in Haefliger's range established in theorem 2.1. Outside of that range it is not known in general if the existence of an equivariant map implies an embedding in this context, but also, to decide whether or not an equivariant map exists within this range has not a definite answer yet. For real projective spaces this is still an open question in a finite number of cases. Solving the known ones can be considered as the starting point in the work of J. Adem, S. Gitler, and I. M. James [1] on immersions. Immersion concept for projective spaces was related to a relatively new sectional category type definition of M. Farber, the so called topological complexity by Farber, Tabachnikov, and Yuzvinsky [11]. It was proved by J. Gonzalez and P. Landweber [15] that the calculation of the symmetric version of topological complexity and the antisymmetric index are equivalent for real projective spaces. This gives most of the known answers on the topic until now, see Theorem 3.8.

Non embedding results for projective spaces give lower bounds for most of the cases in the problems described in the second paragraph of this section. As it is well known, the embedding problem has been studied for long time in topology [19]. We focus our attention on works by M. Mahowald [22] and J. Levine [21] which give the embedding dimension of $\mathbb{R}P^{2^e+1}$ for all e. Using the arguments given in previous paragraph, this gives a complete answer in the said dimensions, the method could be considered as category, immersion, and embedding combination techniques worked out for about fifty years. After the work of Gonzalez-Landweber [15], only the antisymmetric index of $\mathbb{R}P^3$, $\mathbb{R}P^5$ remained unknown (we are still talking about dimensions equal to 2^e+1) as non-immersion results do not give answer for this couple of cases. Studying integral cohomology of $\mathbb{R}P^3 \times \mathbb{R}P^3 - \Delta$ gives the answer for $e = 1$, as shown by J. Gonzalez in [14]. Following this idea it was obtained the integral cohomology ring of $\mathbb{R}P^m \times \mathbb{R}P^m - \Delta$ for all m by J. Gonzalez, P, Landweber and the author in [7], which answered the case $e = 2$. Actually, calculations in the integral cohomology ring allowing us to recover these results, for all e, in a direct way which can be considered as an "antisymmetric index type technique". Here we emphasize on the novelty of this proof rather than on the results in this article.

In Section 2 we recall some classical definitions an well known theorems. Section 3 is about the role of topological complexity on the matter, and in the final section, we describe the idea and write out our new proof which has as a consequence some results concerning real symmetric non-singular bilinear maps also implied by well known (non-)embedding theorems for real projective spaces due to H. Hopf, J. Levine, and M. Mahowald .

2 Some classical results on embeddings, equivariant, and bilinear maps

Let X be a topological space, we denote by $F(X,2)$ the space $X \times X - \Delta$, where Δ is the diagonal in $X \times X$. Of course this is the space of pairs of different points in X, also known as the *configuration space* of two points in X. If now we consider the group of two elements $\mathbb{Z}_2 = \{1, -1\}$ there is an action of this group on $F(X,2)$ given by $(-1)(x,y) = (y,x)$. We refer to this action as the *symmetric action* of $\mathbb{Z}_2$ on $F(X,2)$. The orbit space resulting from the symmetric action is denoted as $B(X,2)$ and is the space of pairs of unordered points in X commonly named the *unordered configuration space* of two points in X.

Let S^{n-1} be the $n-1$ dimensional sphere contained in the Euclidean space $\mathbb{R}^n$, the antipodal action of $\mathbb{Z}_2$ helps to state the classical question, what is the least integer n such that there exits an equivariant map

$$f : F(X,2) \to S^{n-1}$$

with respect to the symmetric and antipodal actions? Such a map is called antisymmetric, and te minimal such n is called *antisymmetric index* of X, and it is denoted by $I_{as}(X)$.

This question is important due to its connection with the embedding problem of manifolds stated by A. Haefliger as follows. Suppose M is a $k-$dimensional smooth manifold embedded in $\mathbb{R}^n$ via $g : M \to \mathbb{R}^n$, then this would define an antisymmetric map $f : F(M,2) \to S^{n-1}$ defined by the formula

$$f(x,y) = \frac{g(x) - g(y)}{||g(x) - g(y)||}.$$

In the work of Haefliger [16] we can find the following result.

Theorem 2.1. *Suppose M is an $m-$dimensional smooth compact manifold. Then, M smoothly embeds in $\mathbb{R}^n$ if there exists an antisymmetric map $F(M,2) \to S^{n-1}$ and $n \geq \frac{3}{2}(m+1)$.*

When M is the real $m-$dimensional projective space $\mathbb{R}P^m$ there are other relations concerning this kind of question coming from a problem in algebra and is about systems of real symmetric bilinear forms.

Consider a homogeneous system of real symmetric bilinear equations

$$\begin{aligned} f_1(x,y) = a^1_{11}x_1y_1 + \cdots + a^1_{rr}x_my_m &= 0 \\ \vdots \quad & \vdots \quad \vdots \\ f_n(x,y) = a^n_{11}x_1y_1 + \cdots + a^n_{mm}x_my_m &= 0. \end{aligned} \tag{1}$$

This means that every $f_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}, i = 1 \ldots n$, is real symmetric $(f_i(x,y) = f_i(y,x)$ for all $x, y \in \mathbb{R}^m)$ bilinear form. A trivial solution to this system is one of the form $(0,y) \in \mathbb{R}^m \times \mathbb{R}^m$. The problem about whether there exists a non-trivial solution to (1) was established by Stiefel, and Hopf gave in [18] a topological context for the problem. In general, the system (1) defines a map

$$\mu : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$$

which is bilinear and symmetric, when the system only has non-trivial solutions we say that μ is non-singular.

Hopf's idea is as follows, suppose μ is bilinear, symmetric, and non-singular, then it defines a map $f : \mathbb{R}P^{m-1} \to S^{n-1}$ by

$$f([x]) = \frac{\mu(x,x)}{||\mu(x,x)||}.$$

It is not difficult to prove that this map is well defined, continuous, and injective, a topological embedding; and for $m > 2$ it would give an embedding of $\mathbb{R}P^{m-1}$ in $\mathbb{R}^{n-1}$. Then the question, given a projective space $\mathbb{R}P^r$, does there exist an Euclidean model for it? arose in this context. Following this line, real and complex polynomial product provides symmetric non-singular bilinear maps for odd and even dimensional cases, giving well known embedding results for real projective spaces.

Theorem 2.2. *Let m be higher than 2. If m is odd, $\mathbb{R}P^{m-1}$ can be embedded in $\mathbb{R}^{2m-2}$. For even m, it can be embedded in $\mathbb{R}^{2m-3}$.*

It is remarkable that Hopf also proved:

Theorem 2.3. *[18, 8] If $m > 2$, it is not possible to embed $\mathbb{R}P^{m-1}$ in $\mathbb{R}^m$.*

Let us call $E(k)$ and $N(k)$ to the least dimension of a Euclidean space where it is possible to embed $\mathbb{R}P^k$ and such that there exists a real symmetric non-singular bilinear map from $\mathbb{R}^k \times \mathbb{R}^k$ respectively. Then, Theorems 2.2 and 2.3 prove at the same time.

Theorem 2.4.

1. $k+2 \leq E(k) \leq N(k+1)-1 \leq \begin{cases} \textit{2k-1 if } k \textit{ is odd,} \\ \textit{2k if } k \textit{ is even.} \end{cases}$

2. *The Fundamental Theorem of Algebra.*

3. $N(3) = 5$, *and* $N(4) = 6$.

4. $E(2) = 4$, *and* $E(3) = 5$.

We remark the following direct inequalities:

$$I_{as}(\mathbb{R}P^k) \leq E(k) \leq N(k+1) - 1. \tag{2}$$

Some numerical conditions can be given on the existence of nonsingular real homogeneous bilinear systems by using different topological methods, a good reference for this matter is [20].

3 Antisymmetric index in relation to (symmetric) topological complexity and (embeddings) immersions of real projective spaces

From the study of robot motion planning problem, a couple of topological concepts, the (Symmetric) Topological Complexity, were extracted by M. Farber and M. Grant [9, 10]. These concepts were formalized using a Lusternik-Schnirelmann Category type notion, namely Sectional Category. We refer to [4] for a general introduction to L-S Category and related topics.

The sectional category of a continuous map $p : E \to B, secat(p)$, is one less than the smallest number of open sets U covering B in such a way that p admits a (continuous) section over each U. Note that we are using a normalized version of the concept for notational considerations in some of the relations appearing in this article.

Consider the following diagram of fibrations,

$$
\begin{tikzcd}
P_2(X) \arrow[d, "p_2"'] & P_1(X) \arrow[l] \arrow[r] \arrow[d, "p_1"'] & P(X) \arrow[d, "p"'] \\
B(X,2) & F(X,2) \arrow[l] \arrow[r] & X \times X,
\end{tikzcd}
\tag{3}
$$

where: $P(X)$ is the free path space $X^{[0,1]}$ with the compact-open topology, p is the end-points evaluation map, and $P_1(X)$ is the subspace of $P(X)$ obtained by removing the free loops on X (p_1 is the restriction of p). The group $\mathbb{Z}/2$ acts freely on both $P_1(X)$ and $F(X,2)$, by running a path backwards in the former (i.e. via $\gamma \mapsto \gamma'$ where $\gamma'(t) = \gamma(1-t)$), and by symmetric action on the latter. Furthermore, p_1 is a $\mathbb{Z}/2$-equivariant map, so that $P_2(X)$ and $B(X,2)$ are the corresponding orbit spaces, and $p_2\colon P_2(X) \to B(X,2)$ denotes the fibration induced by p_1.

The Topological Complexity of X is $TC(X) = secat(p)$ and the Symmetric Topological Complexity of X is defined as

$$TC^S(X) = secat(p_2) + 1.$$

Suppose U is an open subset of $B(X,2)$ such that there exists a continuous section $U \to P_2(X)$, with respect to p_2, let $\widetilde{U}$ be the inverse image of U under the double covering map $F(X,2) \to B(X,2)$, then there exists an equivariant section for p_1 on $\widetilde{U}$. From this fact and asking X to be an ENR, see Lemma 8 in [10], it is proved the following:

Theorem 3.1. *[10]*

$$TC(X) \leq TC^S(X).$$

We will use this result for real projective spaces in connection with embeddings and immersions. For the last one, concepts of Axial and nonsingular maps are remarked now in the context of [11].

A continuous map $g : \mathbb{R}P^m \times \mathbb{R}P^m \to \mathbb{R}P^k, m < k$, is called axial if its restriction on each axis is homotopic to the inclusion $\mathbb{R}P^m \hookrightarrow \mathbb{R}P^k$. A continuous map $f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ is called nonsingular if $f(x,y) = 0$ then $x = 0$ or $y = 0$ and $f(\lambda x, y) = f(x, \lambda y) = \lambda f(x,y)$ for all $x, y \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

In [11] M. Farber, S. Tabachnikov, and S. Yuzvinsky proved that, if $1 < m < k$, there is a bijection between the existence of axial maps $\mathbb{R}P^m \times \mathbb{R}P^m \to \mathbb{R}P^k$, and nonsingular maps $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^{k+1}$. But the main result from the cited work is the next one.

Theorem 3.2. *[11] $TC(\mathbb{R}P^m)$ coincides with the smallest k such that there exists a nonsingular map $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^{k+1}$.*

To relate previous concepts, next classical theorem by J. Adem, S. Gitler, and I. M. James is the key point.

Theorem 3.3. *[1] There exists an immersion $\mathbb{R}P^m \to \mathbb{R}^k$ (where $k > m$) if and only if there exists an axial map $\mathbb{R}P^m \times \mathbb{R}P^m \to \mathbb{R}P^k$.*

Now immersion and categorical concepts can be related in this context.

Theorem 3.4. *[11] For any $m \neq 1, 3, 7$, the number $TC(\mathbb{R}P^m)$ equals the smallest k such that the projective space $\mathbb{R}P^m$ admits an immersion into $\mathbb{R}^k$*

On the other hand, regarding to embeddings and antisymmetric index of projective spaces, the question about whether they are equal or not remains open in general, but a partial answer on this matter, was given by J. González and P. Landweber using (symmetric) topological complexity in [15]. In the following we give a slightly more direct proof of that result plus some cases obtained in [14, 7]. The starting point is the following theorem, which can be considered as the analogue of 3.2 (note that what we call antisymmetric index is named level there).

Theorem 3.5. *[15] For all $m \in \mathbb{N}$*

$$TC^S(\mathbb{R}P^m) = I_{as}(\mathbb{R}P^m).$$

Let us denote by $I_{mm}(m)$ to the smallest dimension of the euclidean space where it is possible to immerse the real projective space $\mathbb{R}P^m$. Then from (2), 3.1, 3.4, and 3.5 we immediately obtain:

Theorem 3.6. *Suppose $m \neq 1, 3, 7$, then*

$$I_{mm}(m) = TC(\mathbb{R}P^m) \leq TC^S(\mathbb{R}P^m) =$$

$$I_{as}(\mathbb{R}P^m) \leq E(\mathbb{R}P^m) \leq N(m+1) - 1.$$

Note that, from previous theorem, if there exists an antisymmetric map $F(\mathbb{R}P^m, 2) \to S^{n-1}$, then there must exits an immersion $\mathbb{R}P^m \to \mathbb{R}^n (m \neq 1, 3, 7)$. The following results on immersions concerns the numerical range established in Theorem 2.1.

Lemma 3.7. *Let m be higher than 15 or equal to $8, 9, 13$. An immersion of $\mathbb{R}P^m \to \mathbb{R}^n$ can exists only if $2n \geq 3(m+1)$.*

Proof. For cases $m \leq 22$ it is just necesary to see the table at the end of [19]. For $m \geq 23$ non immersion results follow from [1, 5] finishing the proof. □

Starting in an idea first observed by A. J. Berrick, S Feder, and S. Gitler in [2], it is possible to determine in almost all of cases whether $E(\mathbb{R}P^m)$ and $I_{as}(\mathbb{R}P^m)$ are equal or not just by applying Theorem 2.1 and Lemma 3.7. Therefore, from 3.5, it is obtained the following result which can be considered as the analogue of 3.4.

Theorem 3.8. *Let $m \in \{1, 2, 3, 4, 5, 8, 9, 13\}$ and $r > 15$. Then:*

1. *$TC^S(\mathbb{R}P^m) = I_{as}(\mathbb{R}P^m) = E(\mathbb{R}P^m)$.*
2. *There exists a smooth embedding of $\mathbb{R}P^m$ in a Euclidean space if and only if there exists a topological embedding in the same Euclidean space.*

Proof. In view of previous paragraph, item 1 follows from Lemma 3.7 and Theorem 2.1 for $m \in \{8, 9, 13\}$ and $m > 15$. Cases $m = 1, 2, 4$ were treated in [15]. $m = 3$ was proved in [14], and $m = 5$ was obtained in [7]. In the final section of this work we will comment on these couple of cases.

To prove item 2, just apply Theorem 2.1 and the second equality in previous item. This kind of argument was remarked by I. M. James in [19] at the end of his introduction. □

Now any numerical result for embedding dimensions in the range established by previous theorem gives the corresponding answer for symmetric topological complexity and antisymmetric index for real projective spaces, we remark the following, because we will recover it from different point of view in final section.

Theorem 3.9.

$$TC^S(\mathbb{R}P^{2^e+1}) = I_{as}(\mathbb{R}P^{2^e+1}) = 2^{e+1} + 1.$$

Proof. From previous theorem it is just a consequence of embedding results for the corresponding projective spaces due to M. Mahowald [22] or J. Levine [21]. □

4 The calculation of a family of antisymmetric indices from the cohomology of reduced symmetric product of Projective spaces.

A classical point of view [17] to calculate the antisymmetric index of projective spaces, as it is argued after the proof of Lemma 4.4, comes from the cohomology of $B(\mathbb{R}P^k, 2)$. This space is also known as the reduced symmetric product of $\mathbb{R}P^k$. The corresponding cohomology ring structure with coefficients in $\mathbb{Z}/2$ was calculated by Handel in [17], see also [6, 12, 13]. These results together with obstruction theory gave some embedding results for projective spaces [17].

Theorem 4.1. *[17]*

$$H^*(B(\mathbb{R}P^{m-1}, 2), \mathbb{Z}/2\mathbb{Z}) \cong \frac{\frac{\mathbb{Z}}{2\mathbb{Z}}[u, v, w]}{J},$$

such that, $dim(u) = dim(v) = 1$, $dim(w) = 2$, and J is the ideal, of the polynomial ring $\frac{\mathbb{Z}}{2\mathbb{Z}}[u, v, w]$, generated by the three elements:

$$\sum_{i=0} \binom{m-1-i}{i} v^{m-1-2i} w^i, \ \sum_{i=0} \binom{m-i}{i} v^{m-2i} w^i, \text{ and } uv + u^2.$$

From the antisymmetric index point of view, J. Gonzalez started studying the problem of calculating the symmetric topological complexity of projective spaces in [14]. He obtained the value of $TC^S(\mathbb{R}P^3)$, proposing the calculation of integral cohomology of $B(\mathbb{R}P^m, 2)$, which was finally achieved basically by the usage of Bokstein spectral sequence and 4.1 in [7].

Theorem 4.2. *[7] Let $m = 2t$, $t \geq 1$. The integral cohomology ring $H^*(B(\mathbb{R}P^m, 2))$ is generated by five classes a_2, b_2, c_3, d_4 and e_{2m-1}, where subscripts denote dimension of the corresponding element, subject only to the relations (where we are omitting the subscripts):*

1. $2a = 2b = 2c = 4d = 0$;
2. $b^2 = ab$;
3. $c^2 = ad$;
4. $\sum \binom{i+j}{j} a^i c d^j = 0$, *where the sum runs over $i, j \geq 0$ with $i + 2j = t - 1$;*

5. $\sum \binom{i+j}{j} a^i b d^j = \begin{cases} 2d^{\frac{t+1}{2}}, & t \text{ odd}, \\ 0, & t \text{ even}, \end{cases}$ *where the sum runs over* $i, j \geq 0$ *with* $i + 2j = t$;
6. $\sum \binom{i+j}{j} a^{i+1} d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i+2j = t$;
7. $\sum \binom{i+j}{j} a^i c d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i+2j = t$;
8. $\sum \binom{i+j}{j} a^i b d^{j+1} = \begin{cases} 2d^{\frac{t+2}{2}}, & t \text{ even}, \\ 0, & t \text{ odd}, \end{cases}$ *where the sum runs over* $i, j \geq 0$ *with* $i + 2j = t - 1$;
9. $d^t = 0$;
10. $e\varepsilon = 0$, *for* $\varepsilon \in \{a, b, c, d, e\}$.

Theorem 4.3. *[7] Let* $m = 2t + 1$, $t \geq 0$. *The integral cohomology ring* $H^*(B(\mathbb{R}P^m, 2))$ *is generated by five classes* a_2, b_2, c_3, d_4 *and* e_m, *where subscripts denote dimension of the corresponding element and we omit them from now on, subject only to the relations:*

1. $2a = 2b = 2c = 4d = 0$;
2. $b^2 = ab$;
3. $c^2 = ad$;
4. $\sum \binom{i+j}{j} a^i b d^j = \begin{cases} 2d^{\frac{t+1}{2}}, & t \text{ odd}, \\ 0, & t \text{ even}, \end{cases}$ *where the sum runs over* $i, j \geq 0$ *with* $i + 2j = t$;
5. $\sum \binom{i+j}{j} a^{i+1} d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i+2j = t$;
6. $\sum \binom{i+j}{j} a^i c d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i+2j = t$;
7. $\sum \binom{i+j}{j} a^i b d^j = \begin{cases} 2d^{\frac{t+2}{2}}, & t \text{ even}, \\ 0, & t \text{ odd}, \end{cases}$ *where the sum runs over* $i, j \geq 0$ *with* $i + 2j = t + 1$;
8. $\sum \binom{i+j}{j} a^{i+1} d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i+2j = t + 1$;

9. $\sum \binom{i+j}{j} a^i c d^j = 0$, *where the sum runs over* $i, j \geq 0$ *with* $i + 2j = t + 1$;

10. $d^{t+1} = 0$;

11. (a) $e^2 = 0$,
    (b) $\mu e = \kappa b^\kappa c d^l$,
    (c) $ce = \eta d^{l+1}$,
    (d) *and* $de = \sum_{i=1}^{l} \binom{t-i}{i-1} a^{t-2i} b c d^i$.
    Here $\mu \in \{a, b\}, t = 2l + \kappa$ *with* $\kappa \in \{0, 1\}$ *and* $\eta = b$, *if* $\kappa = 1$ *whereas* $\eta = 2$ *if* $\kappa = 0$, *except perhaps for* $m = 5$.

Note that this description is presented slightly more explicit than the one given in [7]. The key point is the proof of next lemma, where we use the above algebraic structures in order to obtain the height of a certain relevant element.

Lemma 4.4. *Consider the element* $b \in H^2(B(\mathbb{R}P^m, 2); \mathbb{Z})$ *coming from previous theorems. If* k *is the smallest positive integer such that* $b^k = 0$: *Suppose* $m = 2^e$. *Then*

$$k = 2^e.$$

On the other hand, let $m \in \{2^e + 1, \ldots, 2^{e+1} - 1\}$. *Then*

$$k = 2^e + 1.$$

Proof. First suppose $m = 2^e + 1$; from 4) and 7) in Theorem 4.3 we have

$$a^t b = \binom{t-1}{1} a^{t-2} b d + \binom{t-2}{2} a^{t-4} b d^2 + \cdots \tag{4}$$

and

$$a^{t-1} b d = \binom{t-2}{1} a^{t-3} b d^2 + \binom{t-3}{2} a^{t-5} b d^3 + \cdots + 2 d^{\frac{t+2}{2}}. \tag{5}$$

Note that, in this case, $t = 2^{e-1}$, then using the formula

$$\binom{2^{e-1} - k}{k - 1} \equiv 0 \mod 2 \; \forall \, k$$

it follows that

$$a^{t-1} b d = 2 d^{\frac{t+2}{2}} \text{ and therefore } a^t b d = 0. \tag{6}$$

Multiplying (4) by a^{t-1} and using (6) we get

$$(7)\quad b^{2^e} = b^{m-1} = b^{2t} = a^{2t-1}b = a^{t-1}bd^{t/2} = 2d^{\frac{2t+2}{2}}d^{t/2-1} = 2d^t \neq 0.$$

From dimensional conditions in Theorem 4.3, it is obvious that $b^{2^e+1} = 0$, therefore the proof is complete for this case. Note that this is all what we need to prove Theorem 4.4.

If we consider the case $m = 2^{e+1} - 1$, then $t = 2^e - 1$, and from 4) in 4.3 we get

$$(8)\qquad a^t b = \binom{t-1}{1} a^{t-2}bd + \binom{t-2}{2} a^{t-4}bd^2 + \cdots + 2d^{\frac{t+1}{2}}$$

and from here applying

$$\binom{2^e - k}{k-1} \equiv 0 \mod 2 \;\forall\, k$$

we get

$$(9)\qquad b^{2^e} = b^{t+1} = a^t b = 2d^{\frac{t+1}{2}} \neq 0 \text{ and}$$

$$b^{2^e+1} = b^{t+2} = a^{t+1}b = aa^t b = 2ad^{\frac{t+1}{2}} = 0.$$

It is not difficult to analyze the case $m = 2^e$ in a similar manner, but this time using 4.2, the rest of the cases now follow from these three cases and an inductive argument. $\square$

Now, suppose there exists an antisymmetric map $f : F(\mathbb{R}P^{r-1}, 2) \to S^{n-1}$, passing to the orbit spaces we have the following diagram

$$(10)\qquad \begin{array}{ccc} F(\mathbb{R}P^{r-1},2) & \xrightarrow{f} & S^{n-1} \\ \downarrow & & \downarrow \\ B(\mathbb{R}P^{r-1},2) & \xrightarrow{P(f)} & P^{n-1}. \end{array}$$

The generator z in the second cohomology group of $\mathbb{R}P^{n-1}$ corresponds to the element b_2 appearing in 4.4 under $P(f)^*$, this fact is proved in [7] using group actions on the Stiefel manifold $V_{r,2}$, as is described next.

Consider the dihedral group of order 8

$$(11)\qquad D_8 = \{t, y | t^4 = y^2 = 1, yt = t^3 y\}.$$

D_8 acts freely on the Stiefel manifold $V_{m,2} \subset \mathbb{R}^m \times \mathbb{R}^m$ as $(v_1, v_2)t = (v_2, -v_1), (v_1, v_2)y = (v_1, -v_2)$.

Let H denote the subgroup of D_8 generated by y, yt^2. It is not difficult to prove that $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. If we restrict the action of D_8 to H on $V_{m,2}$, then the orbit space $\frac{V_{m,2}}{H}$, has the homotopy type of $F(\mathbb{R}P^{m-1}, 2)$, and the same assertion olds for $\frac{V_{m,2}}{D_8}$ in comparison to $B(\mathbb{R}P^{m-1}, 2)$. To prove this for $B(\mathbb{R}P^{m-1}, 2)$, just consider the map $g : V_{m,2} \to B(\mathbb{R}P^{m-1}, 2)$ given by $g(v_1, v_2) = ([v_1], [v_2])$. This map clearly passes to the quotient and gives a map $\frac{V_{m,2}}{D_8} \to B(\mathbb{R}P^{m-1}, 2)$. The homotopy inverse is provided by Gram-Schmidt orthogonalization process, applied to generators of a pair of different lines in $\mathbb{R}^m$. Everything is well defined and works right due to identifications on the orbit space, and the argument is similar for $F(\mathbb{R}P^{m-1}, 2)$. In particular this gives a homotopy type fibration

$$V_{m,2} \to B(\mathbb{R}P^{m-1}, 2) \to BD_8, \tag{12}$$

where BD_8 is the classifying space of D_8.

The exact sequence of groups

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \cong H \to D_8 \to \frac{D_8}{H} \cong \mathbb{Z}_2 \to 1,$$

(12), and the double covering fibration

$$F(\mathbb{R}P^{m-1}) \to B(\mathbb{R}P^{m-1}, 2) \to \mathbb{R}P^\infty$$

where used to prove that, in Diagram (10), $P(f)^*(z) = b$. Then a contradiction argument and 4.4, prove the following:

Theorem 4.5.

$$2^{e+1} < I_{as}(\mathbb{R}P^{2^e+1}).$$

Proof. Suppose there exists an antisymmetric map $f : F(\mathbb{R}P^{2^e+1}, 2) \to \mathbb{R}P^{2^{e+1}-1}$, then from previous paragraph $P(f)^*(z) = b$ but $z^{2^e} = 0$ while from 4.4 $b^{2^e} \neq 0$. $\square$

E. Rees proved in [23] that $I_{as}(\mathbb{R}P^6) \leq 9$. Due to the fact $I_{as}(\mathbb{R}P^k) \leq I_{as}(\mathbb{R}P^{k+1})$ for all k, we deduce:

Corollary 4.6.

$$I_{as}(\mathbb{R}P^5) = I_{as}(\mathbb{R}P^6) = 9.$$

It should be remarked that $\mathbb{R}P^6$ is the first real projective space for which, to our knowledge, it is not known whether antisymmetric index coincides to embedding dimension.

As we mentioned, it is not possible to know from non-immersion results if the cases $m = 3, 5$ are in the range of Haefliger's theorem 2.1; they really were obtained and proven to be in the cited range by calculating the integral cohomology of symmetric reduced product of projective spaces in the corresponding dimensions. Due to Equation (2) and Theorem 2.4, the antisymmetric index non-existence point of view can be considered as the method used to obtain a general statement which can be viewed like a straight way to recover Theorem 3.9, part of 3.8, and the corresponding result on real symmetric non singular bilinear maps. It is stated as

Theorem 4.7.

1. $E(2^e + 1) = TC^S(\mathbb{R}P^{2^e+1}) = I_{as}(\mathbb{R}P^{2^e+1}) = 2^{e+1} + 1.$

2. $N(2^e + 2) = 2^{e+1} + 2.$

Proof. Theorem 4.5 gives $2^{e+1} + 1 \leq I_{as}(\mathbb{R}P^{2^e+1})$, which implies

$$2^{e+1} + 1 \leq I_{as}(\mathbb{R}P^{2^e+1}) \leq E(2^e + 1) \leq N(2^e + 2) - 1 \leq 2^{e+1} + 1$$

in view of (2) and Theorem 2.4. $\square$

Acknowledgement

The author would like to thank Professor D. Juan for his useful suggestions on an earlier version of this article, and to Professor J. Gonzalez for the careful reading of the final version and the corresponding comments.

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