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The stability theorem of persistent homology

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Abstract

Persistence modules – an important tool for understanding geometric properties of data and the central objects of study in persistent homology – are collections of vector spaces indexed by the real numbers together with linear maps satisfying certain basic properties. Persistence modules appear naturally as families of homology groups associated to filtrations of topological spaces. Two equivalent ways to represent a persistence module are by its persistence diagram – a multiset of points in the Euclidean plane – and by its barcode – a multiset of real intervals. Under certain assumptions (commonly satisfied in practice), these representations exist and are unique. One of the main results in the theory of persistence is the *Stability Theorem*, which asserts that small perturbations of a persistence module result in small perturbations of its persistence diagram and barcode. In this paper, we review the evolution of this theorem, with emphasis on the results appearing in *The Structure and Stability of Persistence Modules* (Chazal et al., 2012) and *Induced Matchings and the Algebraic Stability of Persistence Barcodes* (Bauer and Lesnick, 2015).

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Contents

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1 Introduction

Persistent homology, a topological tool for analyzing the global, nonlinear, geometric features of data, is the primary object of study in topological data analysis. Frequently in science and engineering, data can be naturally represented as a **filtration**: namely a collection of topological spaces X_t with $t \in \mathbb{R}$ such that $X_s \subset X_t$ whenever $s \leq t$, of which the sublevel set filtration $X_t := f^{-1}((-\infty, t])$ for a continous function $f : X \to \mathbb{R}$ is a common example. To gain insight into the structure of such data, one can apply the homology functor F which takes a topological space X to the vector space $H_n(X, \mathbf{k})$ (with n a fixed nonnegative integer and $\mathbf{k}$ a fixed field), and continuous maps $f : X \to Y$, to the induced linear maps $f_* : H_n(X, \mathbf{k}) \to H_n(Y, \mathbf{k})$. By the functoriality of F, this yields a collection $M_t := H_n(X_t, \mathbf{k})$ of vector spaces indexed by $\mathbb{R}$ together with linear maps $\varphi_M(s,t) := \iota(s,t)_* : M_s \to M_t$, $s \leq t$ induced by the inclusion maps $\iota(s,t) : X_s \hookrightarrow X_t$ such that

(i) $\varphi_M(s,t) \circ \varphi_M(r,s) = \varphi_M(r,t)$ whenever $r \leq s \leq t$, and

(ii) $\varphi_M(t,t) = \mathbb{1}_{M_t}$,

where $\mathbb{1}_{M_t} : M_t \to M_t$ denotes the identity map. A collection of vector spaces M_t, $t \in \mathbb{R}$ with linear maps $\varphi_M(s,t)$, $s \leq t$ satisfying (i) and (ii) is called a **persistence module**.

In [8], Edelsbrunner, Letscher and Zomorodian introduce the **persis−tence diagram dgm**(M) – a multiset of points in the Euclidean plane which encodes information about any given persistence module M satisfying certain finiteness (or "tameness") conditions – and provide a fast algorithm for computing the persistence diagram when the underlying topological spaces are finite simplicial complexes in $\mathbb{R}^3$. In [3], Carlsson, Collins, Guibas and Zomorodian introduce the **barcode** $\mathcal{B}_M$ – a multiset of intervals ("bars") which encodes nearly identical (but slightly finer) information.

When analyzing real-world data, uncertainties in the data provided are bound to appear, whether this is due to measurement errors, discretization errors or other sources. Therefore, it is essential to distinguish inherent topological features of the data from noise. One way to quantify these errors is via **$\boldsymbol{\delta}-$matchings** – a bijection between two multisets in the Euclidean plane such that the distance between two corresponding points is at most $\delta > 0$. The **bottleneck distance** $d_B(\cdot, \cdot)$ is the pseudometric on persistence diagrams defined to be the infimum over all $\delta > 0$ such that there exists a δ-matching of the persistence diagrams. A pseudometric $d_B(\cdot, \cdot)$ can be defined similarly for barcodes.

In [6], Cohen-Steiner, Edelsbrunner and Harer study sublevel-set filtrations X^f of real-valued continuous functions $f : X \to \mathbb{R}$ on topological spaces. They show that, under certain mild finiteness (or "tameness") assumptions, small perturbations of functions produce small perturbations in the persistence diagram of the persistence modules with $M_t^f = H_n(X_t^f, \mathbf{k})$; specifically, the bottleneck distance between the persistence diagrams of two functions f and g is bounded above by their distance in the infinity norm: $d_B\big(\mathbf{dgm}(M^f), \mathbf{dgm}(M^g)\big) \leq \|f - g\|_\infty$. This is the first incarnation of the **Stability Theorem** for persistence modules.

A **$\boldsymbol{\delta}-$interleaving** is an approximate isomorphism of persistence modules, with the error in the approximation quantified by δ. The **interleaving distance** $d_I(\cdot, \cdot)$ is the pseudometric on persistence modules defined to be the infimum over all $\delta > 0$ such that there exists a δ-interleaving of the persistence modules. By replacing the ∞-norm by the interleaving distance, Chazal, Steiner, Glisse, Guibas and Oudot show in [4] that the Stability Theorem of [6] holds for persistence modules M and N satisfying similar tameness conditions: $d_B(\mathbf{dgm}(M), \mathbf{dgm}(N)) \leq d_I(M, N)$. This version of stability is called the **Algebraic Stability Theorem**, as it is a purely algebraic statement. This inequality is in fact an equality – $d_B(\mathbf{dgm}(M), \mathbf{dgm}(N)) = d_I(M, N)$ – a result known as the **Isometry Theorem**, which is first proven by Lesnick for so-called pointwise finite-dimensional (PFD) persistence modules in a

2011 version of [9]. The inequality $d_B(\mathbf{dgm}(M), \mathbf{dgm}(N)) \geq d_I(M, N)$ is called the **Converse Stability Theorem**.

In *The Structure and Stability of Persistence Modules* [5], Chazal, de Silva, Glisse, and Oudot provide a comprehensive overview of the theory of persistence. Furthermore, they introduce a more general tameness condition, called q-tameness, and prove the Algebraic Stability Theorem and the Converse Stability Theorem for q-tame persistence modules using rectangle measures, functions defined on rectangles in the Euclidean plane with properties analogous to measures.

In *Induced Matchings and the Algebraic Stability of Persistence Barcodes* [2], Bauer and Lesnick prove a stronger, explicit version of the Algebraic Stability Theorem: given a δ-interleaving of persistence modules of finite dimensional vector spaces, they provide an explicit δ-matching of the respective barcodes.

The current paper will survey these last two papers, focusing on the background and results necessary to prove the Algebraic Stability Theorem (Theorem 3.5.2) and the Converse Stability Theorem (Theorem 2.3.14).

2 The Structure and Stability of Persistence Modules

The following section draws primarily from the exposition of Chazal, de Silva, Glisse and Oudot in *The Structure and Stability of Persistence Modules*[5]. Some of the notation comes from Bauer and Lesnick[2], and several theorems and definitions first appear in earlier works such as Cohen-Steiner et al.[8] (in which the definition of the persistence diagram first appears).

All vector spaces shall be over a fixed field $\mathbf{k}$.

Definition 2.0.1. Recall from the introduction that a **persistence module** M is a collection of vector spaces M_t for $t \in \mathbb{R}$ together with a collection of linear maps $\varphi_M(s,t) : M_s \to M_t$ for every $s \leq t$ – called **transition maps** – satisfying the composition law

$$\varphi_M(s,t) \circ \varphi_M(r,s) = \varphi_M(r,t)$$

whenever $r \leq s \leq t$ and such that $\varphi_M(t,t) : M_t \to M_t$ is the identity for every $t \in \mathbb{R}$. M is said to be **pointwise finite-dimensional** (**PFD**) if each vector space M_t is finite-dimensional.

Let $\mathbf{R} = (\mathbb{R}, \leq)$ be the category with objects $t \in \mathbb{R}$ and a unique morphism $s \to t$ whenever $s \leq t$. This leads to an equivalent definition of a persistence module:

Definition 2.0.2. A persistence module M is a functor $M : \mathbf{R} \to \mathbf{Vect}$ from the category $\mathbf{R} = (\mathbb{R}, \leq)$ to the category $\mathbf{Vect}$ of vector spaces over $\mathbf{k}$. A PFD persistence module M over $\mathbb{R}$ is a functor $M : \mathbf{R} \to \mathbf{vect}$ from the category $\mathbf{R} = (\mathbb{R}, \leq)$ to the category $\mathbf{vect}$ of *finite-dimensional* vector spaces over $\mathbf{k}$.

Definition 2.0.3. A **morphism** $f : M \to N$ between two persistence modules M and N is a collection of linear maps

$$f_t : M_t \to N_t, \quad t \in \mathbb{R}$$

such that the following diagram commutes whenever $s \leq t$:

$$\begin{array}{ccc} M_s & \xrightarrow{\varphi_M(s,t)} & M_t \\ \downarrow{\scriptstyle f_s} & & \downarrow{\scriptstyle f_t} \\ N_s & \xrightarrow{\varphi_N(s,t)} & N_t \end{array}$$

The composition of morphisms $f : M \to N$ and $g : N \to P$ is the pointwise composition $(g \circ f)_t = g_t \circ f_t$. This is clearly associative with identity morphism $\mathbb{1}_M : M \to M$ equal pointwise to the identity $(\mathbb{1}_M)_t = \mathbb{1}_{M_t} : M_t \to M_t$.

Remark 2.0.4. In categorical terminology, a morphism $f : M \to N$ is a natural transformation between the functors $M, N : \mathbf{R} \to \mathbf{Vect}$.

Remark 2.0.5. The collection of all persistence models together with morphisms as defined above forms a category, denoted by $\mathbf{Vect}^{\mathbf{R}}$. This category contains kernels, images and direct sums (categorical coproducts). We construct the direct sum of persistence modules below.

Definition 2.0.6. Let M and N be persistence modules. The **direct sum** of M and N, denoted by $P = M \oplus N$, is the persistence module with vector spaces

$$P_t = M_t \oplus N_t$$

and linear maps

$$\varphi_P(s,t) = \varphi_M(s,t) \oplus \varphi_N(s,t)$$

from P_s to P_t whenever $s \leq t$.

For a collection of persistence modules $\{P_k \mid k \in \mathcal{K}\}$, the direct sum $P = \bigoplus_{k \in \mathcal{K}} P_k$ is defined analogously.

Remark 2.0.7. The reader may check that projection maps $\pi_M : M \oplus N \to M$ and $\pi_N : M \oplus N \to N$, defined pointwise by $(\pi_M)_t(v \oplus w) = v$ and $(\pi_N)_t(v \oplus w) = w$ for $v \in M_t$, $w \in N_t$, are morphisms of persistence modules. Similarly, the pointwise inclusions $\iota_M : M \to M \oplus N$ and $\iota_N : N \to M \oplus N$ are morphisms (in fact these are the canonical injections from the definition of a categorical coproduct).

2.1 Decomposable Persistence Modules

We say that $I \subset \mathbb{R}$ is an **interval** if I is non-empty and $r, t \in I$ implies $s \in I$ whenever $r \leq s \leq t$.

Definition 2.1.1. Let $I \subset \mathbb{R}$ be an interval. The **interval module** corresponding to I, denoted by $C(I)$, is the persistence module with vector spaces

$$C(I)_t = \begin{cases} \mathbf{k} & \text{if } t \in I \\ 0 & \text{otherwise,} \end{cases}$$

and transition maps

$$\varphi_{C(I)}(s,t) = \begin{cases} \mathbb{1}_{\mathbf{k}} & \text{if } s, t \in I \\ 0 & \text{otherwise} \end{cases}$$

from $C(I)_s$ to $C(I)_t$ whenever $s \leq t$. Given a persistence module M, we will refer to a submodule $N \subset M$ which is isomorphic to $C(I)$ as an **interval submodule** of M, or more specifically an **I-submodule** of M.

A persistence module M is said to be **decomposable** if M is isomorphic to a direct sum of interval modules:

$$M \cong \bigoplus_{k \in \mathcal{K}} C(I_k)$$

where each $I_k \subset \mathbb{R}$ is an interval.

In general, there may be repeated intervals in a direct sum decomposition, i.e. intervals $I_k = I_{k'}$ with $k \neq k'$. It is therefore convenient to introduce the notion of a multiset:

Definition 2.1.2. A **multiset** is a pair $\mathbf{A} = (S, m)$ where S is a set and $m : S \to \mathbb{N}$ is a multiplicity function from S to the positive integers $\mathbb{N}$. Intuitively, $m(s)$ is the number of copies of $s \in S$ appearing in the multiset $\mathbf{A}$. The **representation** $\mathrm{Rep}(\mathbf{A})$ of the multiset $\mathbf{A}$ is the set

$$\mathrm{Rep}(\mathbf{A}) = \{(s, n) \mid s \in S,\ n \in \mathbb{N},\ n \leq m(s), \}.$$

Remark 2.1.3. More generally, we may allow $m : S \to \mathbf{Card}$ to take values in the proper class of cardinal numbers.

Throughout this paper, we often work with a collection of elements I_k of some set indexed by $k \in \mathcal{K}$ (for example, see the decomposable module M in Definition 2.1.1). Let $\mathbf{A} = (S, m)$ be the multiset with

$$S = \{I_k \mid k \in \mathcal{K}\}$$

and multiplicity function

$$m(I) = \big|\{k \in \mathcal{K} \mid I_k = I\}\big|, \quad I \in S.$$

In this case, we shall use double curly brackets

$$\{\!\{I_k \mid k \in \mathcal{K}\}\!\} := \mathrm{Rep}(\mathbf{A})$$

to denote the representation of $\mathbf{A}$.

Given a decomposition of a persistence module into a direct sum of interval modules, it is natural to ask if this decomposition is unique. Theorem 2.1.4 answers this question affirmatively:

Theorem 2.1.4 (Unique Decomposition Theorem)**.** *Let*

$$M = \bigoplus_{k \in \mathcal{K}} P^{I_k} = \bigoplus_{l \in \mathcal{L}} Q^{J_l}$$

where the P^{I_k}, $k \in \mathcal{K}$ and Q^{J_l}, $l \in \mathcal{L}$ are respectively I_k- and J_l-submodules of M. Then $\{\!\{I_k \mid k \in \mathcal{K}\}\!\} = \{\!\{J_l \mid l \in \mathcal{L}\}\!\}$.

As observed in Chazal et al.[5](Theorem 1.3), this is a corollary of Azumaya[1](Theorem 1). We present another proof below.

Let $s \in \mathbb{R}$ and let $I \subset \mathbb{R}$ be an interval. We say that $s > I$ if $s > t$ for every $t \in I$, and $s < I$ if $s < t$ for every $t \in I$.

Lemma 2.1.5. *Let $I, J \subset \mathbb{R}$ be intervals.*

(i) Every morphism $f : C(I) \to C(J)$ is of the form

$$f_t = \begin{cases} a \cdot \mathbb{1}_{\mathbf{k}} & \text{if } t \in I \cap J \\ 0 & \text{otherwise} \end{cases}$$

for some $a \in \mathbf{k}$.

(ii) If there exist nonzero morphisms $f : C(I) \to C(J)$ and $g : C(J) \to C(I)$, then $I = J$.

Proof. If $t \notin I \cap J$ then either $C(I)_t = 0$ or $C(J)_t = 0$, so clearly $f_t = 0$. If $t \in I \cap J$, then since the only linear maps $\mathbf{k} \to \mathbf{k}$ are multiplication by a scalar, $f_t = a_t \cdot \mathbb{1}_{\mathbf{k}}$ for some $a_t \in \mathbf{k}$. If $s, t \in I \cap J$ with $s \leq t$ then $a_s = a_t = a$ by the commutativity of the following diagram:

$$\begin{tikzcd} C(I)_s \arrow[r, "\mathbb{1}_{\mathbf{k}}"] \arrow[d, "a_s \cdot \mathbb{1}_{\mathbf{k}}"] & C(I)_t \arrow[d, "a_t \cdot \mathbb{1}_{\mathbf{k}}"] \\ C(J)_s \arrow[r, "\mathbb{1}_{\mathbf{k}}"] & C(J)_t \end{tikzcd}$$

This proves Part (i).

For Part (ii), use Part (i) to write

$$f_t = \begin{cases} a \cdot \mathbb{1}_{\mathbf{k}} & \text{if } t \in I \cap J \\ 0 & \text{otherwise} \end{cases}$$

and

$$g_t = \begin{cases} b \cdot \mathbb{1}_{\mathbf{k}} & \text{if } t \in I \cap J \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $I \neq J$, and suppose without loss of generality that there exists some $t \in I \backslash J$. Then either $t > J$ or $t < J$. If $t > J$ then $s < t$ for every $s \in I \cap J$. By the commutativity of the diagram

$$\begin{tikzcd} C(I)_s \arrow[r, "\mathbb{1}_{\mathbf{k}}"] \arrow[d, "a \cdot \mathbb{1}_{\mathbf{k}}"] & C(I)_t \arrow[d, "0"] \\ C(J)_s \arrow[r, "0"] \arrow[d, "b \cdot \mathbb{1}_{\mathbf{k}}"] & C(J)_t = 0 \arrow[d, "0"] \\ C(I)_s \arrow[r, "\mathbb{1}_{\mathbf{k}}"] & C(I)_t \end{tikzcd}$$

we must have $ab = 0$, which is equivalent to $f = 0$ or $g = 0$. The case when $t < J$ is similar. $\square$

Proof of the Unique Decomposition Theorem. Fix any interval K of $\mathbb{R}$ and consider the submodules P and Q which are direct sums respectively of all P^{I_k} such that $I_k = K$, or all Q^{J_l} such that $J_l = K$. It suffices to show that $\dim(P)_t = \dim(Q)_t$ for all $t \in \mathbb{R}$.

Let $V = \oplus_{k\in\mathcal{K},I_k\neq K} P^{I_k}$ so $M = V \oplus P$, and similarly define W so $M = W \oplus Q$. Let π_V, π_P, π_W and π_Q be projections onto the direct summands V, P, W and Q respectively. While it is not in general true that P=Q, we will show that projection of either P or Q into the other is a monomorphism. The identity on M can be decomposed as $\mathbb{1}_M = \pi_W + \pi_Q$, and so $\mathbb{1}_P = \pi_P|_P = \pi_P \circ (\pi_W + \pi_Q)|_P$. Let $g := \pi_P \circ \pi_W$. We claim $P \subset \ker g$ so $\mathbb{1}_P = \pi_P \circ \pi_Q|_P$. Indeed, π_W can be further decomposed as a sum of projection morphisms into each of the interval submodules Q^{J_l}, $l \in \mathcal{L}, J_l \neq K$, and so for any $I_k = K$, $g|_{P^{I_k}}$ is a sum of morphisms from the K-submodule P^{I_k} to itself, each factoring through a J_l-interval submodule, $l \in \mathcal{L}, J_l \neq K$; by Lemma 10(ii) such morphisms are trivial. Thus, $\mathbb{1}_P = \pi_P \circ \pi_Q|_P$ and it follows that $\dim P_t = \dim \pi_Q(P)_t \leq \dim Q_t$, $\forall t \in \mathbb{R}$. The analogous argument with roles of P and Q reversed gives $\dim Q_t \leq \dim P_t$. $\square$

Remark 2.1.6. Theorem 2.1.4 does not guarantee that all persistence modules are decomposable, but merely that if a decomposition exists, it is unique. However, the following theorem guarantees that a large class of persistence modules are decomposable:

Theorem 2.1.7. *Every pointwise finite-dimensional (PFD) persistence module is decomposable.*

The proof, which is beyond the scope of this paper, appears in [7] (Theorem 1.1).

2.2 Interleaving

Let M and N be persistence modules, and let $\delta \geq 0$. Following the notation of Bauer and Lesnick in [2], we define $M(\delta)$ to be the persistence module with vector spaces $M(\delta)_t = M_{t+\delta}$ and transition maps $\varphi_{M(\delta)}(s,t) = \varphi_M(s+\delta, t+\delta)$ for $s \leq t$. Given a morphism $f : M \to N$, we define the morphism $f(\delta) : M(\delta) \to N(\delta)$ by $f(\delta)_t = f_{t+\delta}$.

Definition 2.2.1. The $\boldsymbol{\delta}$**−shift functor** $(\delta)(\cdot) : \mathbf{Vect}^{\mathbf{R}} \to \mathbf{Vect}^{\mathbf{R}}$ is the functor sending a persistence module M to $M(\delta)$ and a morphism $f : M \to N$ to $f(\delta) : M(\delta) \to N(\delta)$.

Remark 2.2.2. For $\delta > 0$ and $a, b \in \mathbb{R}$ with $a < b$, we have $C\big([a,b]\big)(\delta) = C\big([a-\delta, b-\delta]\big)$. More generally, if $M \cong \bigoplus_{k\in\mathcal{K}} C(I_k)$ then $M(\delta) \cong \bigoplus_{k\in\mathcal{K}} C(I_k(\delta))$, where $I_k(\delta)$ is the interval I_k with its endpoints shifted to the left by δ.

Definition 2.2.3. Let M be a persistence module. For any real number $\epsilon \geq 0$, the transition maps of M define a canonical morphism $\varphi_M^\epsilon : M \to M(\epsilon)$ called the $\boldsymbol{\epsilon}-$ **transition morphism**:

$$(\varphi_M^\epsilon)_t = \varphi_M(t, t+\epsilon)$$

for every $t \in \mathbb{R}$.

Remark 2.2.4. φ_M^ϵ is a morphism because the diagram

$$\begin{tikzcd} M_s \arrow[r, "\varphi_M(s,t)"] \arrow[d, "\varphi_M(s,s+\epsilon)"] & M_t \arrow[d, "\varphi_M(t,t+\epsilon)"] \\ M_{s+\epsilon} \arrow[r, "\varphi_M(s+\epsilon,t+\epsilon)"] & M_{t+\epsilon} \end{tikzcd}$$

commutes whenever $s \leq t$ by the definition of a persistence module.

Definition 2.2.5. Two persistence modules M and N are said to be $\boldsymbol{\delta}$ $-$**interleaved** if there are morphisms $f : M \to N(\delta)$ and $g : N \to M(\delta)$ such that $g(\delta) \circ f = \varphi_M^{2\delta}$ and $f(\delta) \circ g = \varphi_N^{2\delta}$. More explicitly, we require that the diagrams

$$\begin{tikzcd} M_s \arrow[r, "\varphi_M(s,t)"] \arrow[d, "f_s"] & M_t \arrow[d, "f_t"] \\ N_{s+\delta} \arrow[r, "\varphi_N(s+\delta,t+\delta)"] & N_{t+\delta} \end{tikzcd} \qquad \begin{tikzcd} M_{s-\delta} \arrow[rr, "\varphi_M(s-\delta,s+\delta)"] \arrow[dr, "f_{s-\delta}"] & & M_{s+\delta} \\ & N_s \arrow[ur, "g_s"] & \end{tikzcd}$$

$$\begin{tikzcd} N_s \arrow[r, "\varphi_N(s,t)"] \arrow[d, "g_s"] & N_t \arrow[d, "g_t"] \\ M_{s+\delta} \arrow[r, "\varphi_M(s+\delta,t+\delta)"] & M_{t+\delta} \end{tikzcd} \qquad \begin{tikzcd} N_{s-\delta} \arrow[rr, "\varphi_N(s-\delta,s+\delta)"] \arrow[dr, "g_{s-\delta}"] & & N_{s+\delta} \\ & M_s \arrow[ur, "f_s"] & \end{tikzcd}$$

commute whenever $s < t$. f and g are then called $\boldsymbol{\delta}-$**interleavings**, and f is said to be a $\boldsymbol{\delta}-$**inverse** of g (and vice-versa).

Remark 2.2.6. A δ-inverse g is not unique in general. For example, let M and N be any two persistence module such that $\varphi_M(t, t+2\delta) = 0 : M_t \to M_{t+2\delta}$ and $\varphi_N(t, t+2\delta) = 0 : N_t \to N_{t+2\delta}$ for all $t \in \mathbb{R}$ – for instance, $M = C([0,\delta))$ and $N = C([p, p+\delta_0])$ with $p \in \mathbb{R}$ and $0 < \delta_0 < \delta$. Then M and N are δ-interleaved, where the δ-interleaving f in Definition 2.2.5 can be taken to be the zero morphism $f = 0 : M \to N(\delta)$. We note that any morphism $g : N \to M(\delta)$ is a δ-inverse of f.

Remark 2.2.7. If two persistence modules M and N are δ-interleaved, then they are ϵ-interleaved for every $\epsilon \geq \delta$. Indeed, if $f : M \to N(\delta)$ is a δ-interleaving with δ-inverse $g : N \to M(\delta)$, then the composition $f(\epsilon - \delta) \circ \varphi_M^{\epsilon-\delta}$ is an ϵ-interleaving with ϵ-inverse $g(\epsilon - \delta) \circ \varphi_N^{\epsilon-\delta}$, which can be easily seen by observing that $f(\epsilon - \delta) \circ \varphi_M^{\epsilon-\delta} = \varphi_N^{\epsilon-\delta}(\delta) \circ f$ and $g(\epsilon - \delta) \circ \varphi_N^{\epsilon-\delta} = \varphi_M^{\epsilon-\delta}(\delta) \circ g$.

Remark 2.2.8. Let M, N and P be persistence modules. If M and N are δ_1-interleaved, and N and P are δ_2-interleaved, then M and P are $(\delta_1 + \delta_2)$-interleaved. Indeed, if $f_1 : M \to N(\delta_1)$ is a δ_1-interleaving with δ_1-inverse $g_1 : N \to M(\delta_1)$, and if $f_2 : N \to P(\delta_2)$ is a δ_2-interleaving with δ_2-inverse $g_2 : P \to N(\delta_2)$, then the composition $f = f_2(\delta_1) \circ f_1 : M \to P(\delta_1 + \delta_2)$ is a $(\delta_1 + \delta_2)$-interleaving with $(\delta_1 + \delta_2)$-inverse $g = g_1(\delta_2) \circ g_2 : P \to M(\delta_1 + \delta_2)$.

Definition 2.2.9. The **interleaving distance** $d_I(\cdot, \cdot)$ between two persistence modules M and N is the infimum over all non-negative real numbers such that M and N are δ-interleaved:

$$d_I(M, N) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-interleaving between } M \text{ and } N\}$$

Lemma 2.2.10. *The interleaving distance satisfies the triangle inequality: for any three persistence modules M, N and P, we have*

$$d_I(M, P) \leq d_I(M, N) + d_I(N, P).$$

Proof. By Remark 2.2.8, if M and N are δ_1-interleaved and N and P are δ_2-interleaved, then M and P are $(\delta_1 + \delta_2)$-interleaved. The result follows by taking the infimum over all such δ_1 and δ_2. □

Proposition 2.2.11. *Let $M = \bigoplus_{j \in \mathcal{J}} M_j$ and let $N = \bigoplus_{j \in \mathcal{J}} N_j$ (with the same indexing set $\mathcal{J}$). Then*

$$d_I(M, N) \leq \sup_{j \in \mathcal{J}} d_I(M_j, N_j).$$

Proof. If for every $j \in \mathcal{J}$ there exists an ϵ-interleaving $f_j : M_j \to N_j(\epsilon)$, then $f = \bigoplus_{j \in \mathcal{J}} f_j$ is an ϵ-interleaving of M and N, and so $d_I(M, N) \leq \epsilon$. Thus by Remark 2.2.7, any upper bound of the interleaving distances $d_I(M_j, N_j)$ must then be an upper bound of $d_I(M, N)$; in particular, this applies to the supremum of the $d_I(M_j, N_j)$. □

2.3 Decorated Real Numbers and Persistence Diagrams

We now introduce the notion of decorated real numbers, which simplify interval notation and play an important role in the approach of [5] and [2].

Definition 2.3.1. The **decorated real numbers**, denoted by $\mathbb{D}$, are the collection of ordered pairs $(p, \pm)$ with $p \in \mathbb{R}$ – which we shall denote henceforth by $p^{\pm}$, or p^* if the "decoration" $* \in \{-,+\}$ is unspecified – together with $\pm\infty$. Ordering $\{-,+\}$ by setting $- < +$, we endow $\mathbb{D}$ with the lexicographic order, with $-\infty$ and $+\infty$ the minimum and maximum elements of the order respectively. Explicitly, $p^* < q^{*'}$ if and only if either $p < q$, or $p = q$ and $* = -$, $*' = +$.

Remark 2.3.2. The **extended real numbers**, denoted by $\overline{\mathbb{R}}$, are the real numbers $\mathbb{R}$ with the standard order together with a maximal element ∞ and a minimal element $-\infty$ (i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$). The extended real numbers $\overline{\mathbb{R}}$ can be obtained from the decorated real numbers $\mathbb{D}$ by "forgetting" the decorations – namely by identifying p^+ and p^- with p ($p \in \mathbb{R}$).

Remark 2.3.3. We can identify ordered pairs $(p^*, q^*) \in \mathbb{D} \times \mathbb{D}$ with intervals of real numbers whenever $p^* < q^*$ as in the following table:

	q^-	q^+	∞
p^-	$[p, q)$	$[p, q]$	$[p, \infty)$
p^+	(p, q)	$(p, q]$	(p, ∞)
$-\infty$	$(-\infty, q)$	$(-\infty, q]$	$(-\infty, \infty)$

When we refer to the interval corresponding to the ordered pair (p^*, q^*), we shall use the notation $\langle p^*, q^* \rangle$.

Definition 2.3.4. Let M be a decomposable persistence module with

$$M \cong \bigoplus_{k \in \mathcal{K}} C\big(\langle p_k^*, q_k^* \rangle\big).$$

The **decorated persistence diagram** of M is the multiset of pairs of decorated real numbers

$$\mathbf{Dgm}(M) = \{\!\!\{ (p_k^*, q_k^*) \mid k \in \mathcal{K} \}\!\!\}.$$

The **undecorated persistence diagram** of M is the multiset of pairs of (undecorated, extended) real numbers

$$\mathbf{dgm}(M) = \{\!\!\{ (p_k, q_k) \mid k \in \mathcal{K} \}\!\!\}.$$

Remark 2.3.5. By Theorem 2.1.4, the (un)decorated persistence diagram of a decomposable persistence module does not depend on the choice of decomposition.

Definition 2.3.6. A function σ is said to be a **partial matching** between sets A and B if σ is a bijection between a subset $A' \subset A$ and a subset $B' \subset B$. In other words, we say that σ is a partial matching between A and B if σ is a bijection with $\mathrm{dom}(\sigma) \subset A$ and $\mathrm{im}(\sigma) \subset B$. We write $\sigma : A \nrightarrow B$ to mean "σ is a partial matching between A and B." We say that a pair of points $a \in A$ and $b \in B$ are **matched** if $a \in \mathrm{dom}(\sigma)$, $b \in \mathrm{im}(\sigma)$ and $\sigma(a) = b$.

The composition of a partial matching σ_1 between A and B and a partial matching σ_2 between B and C is the partial matching

$$\sigma_2 \circ \sigma_1 = \{(a, c) \mid \exists b \in B \text{ such that } (a, b) \in \sigma_1 \text{ and } (b, c) \in \sigma_2\}.$$

Remark 2.3.7. In [2] (Bauer and Lesnick), the authors refer to partial matchings simply as matchings. We use terminology partial matching which appears in [5] (Chazal et al.) to emphasize that only some of the elements in partially matched sets are matched.

We shall soon define a distance between subsets of the extended plane $\overline{\mathbb{R}}^2$ using partial matchings, but first we must choose a point-to-point distance. Let $d^\infty(\cdot, \cdot)$ denote the **infinity norm** on the extended plane, defined by

$$d^\infty\big((p, q), (r, s)\big) = \max\{|r - p|, |s - q|\}.$$

Here we extend $|r - p|$ to allow for p or r to be infinite by setting $|r - p| = \infty$ if one of p or r is infinite and $p \neq r$, and $|r - p| = 0$ if $p = r$ – for instance, we have $|\infty - 2| = |\infty - (-\infty)| = \infty$, but $|\infty - \infty| = 0$.

If $(p, q) \in \overline{\mathbb{R}}^2$ and $S \subset \overline{\mathbb{R}}^2$, let

$$d^\infty\big((p, q), S\big) = \inf_{(r,s) \in S} d^\infty\big((p, q), (r, s)\big).$$

In particular, letting

$$\Delta = \{(p, q) \in \mathbb{R}^2 \mid q = p\},$$

denote the diagonal, it is easy to see that

$$d^\infty\big((p, q), \Delta\big) = \frac{1}{2}|q - p|.$$

Definition 2.3.8. A partial matching σ between two subsets A and B of $\mathbb{R}^2$ is called a $\boldsymbol{\delta}$**−matching** if all of the following statements are true:

- if $a \in A$ and $b \in B$ are matched in σ (i.e. $\sigma(a) = b$) then $d^\infty(a, b) \leq \delta$;
- if $a \in A$ is unmatched in σ (i.e. $a \notin \mathrm{dom}(\sigma)$) then $d^\infty(a, \Delta) \leq \delta$;
- if $b \in B$ is unmatched in σ (i.e. $b \notin \mathrm{im}(\sigma)$) then $d^\infty(b, \Delta) \leq \delta$.

We say that A and B are $\boldsymbol{\delta}$**−matched** if there exists a δ-matching between A and B.

Remark 2.3.9. Let $A, B, C \subset \mathbb{R}^2$. If A and B are δ_1-matched, and B and C are δ_2-matched, then A and C are $(\delta_1 + \delta_2)$-matched. Indeed, let $\sigma_1 : A \nrightarrow B$ be a δ_1-matching and let $\sigma_2 : B \nrightarrow C$ be a δ_2-matching. We verify that $\sigma = \sigma_2 \circ \sigma_1 : A \nrightarrow C$ is a $(\delta_1 + \delta_2)$-matching:

- If $a \in A$ and $c \in C$ are matched in σ, then there is some $b \in B$ such that $(a, b) \in \sigma_1$ and $(b, c) \in \sigma_2$, and so
$$d^\infty(a, c) \leq d^\infty(a, b) + d^\infty(b, c) \leq \delta_1 + \delta_2.$$
- If $a \in A$ is unmatched in σ, then either a is unmatched in σ_1, in which case
$$d^\infty(a, \Delta) \leq \delta_1 \leq \delta_1 + \delta_2;$$
or a is matched in σ_1 to some $b \in B$ which is unmatched in σ_2, in which case
$$d^\infty(a, \Delta) \leq d^\infty(a, b) + d^\infty(b, \Delta) \leq \delta_1 + \delta_2.$$
- The case of an unmatched point $c \in C$ is similar to the previous case.

Definition 2.3.10. The **bottleneck distance** $d_B(\cdot, \cdot)$ between two subsets A and B of $\mathbb{R}^2$ is the infimum over all non-negative real numbers such that A and B are δ-matched:

$$d_B(A, B) = \inf\{\delta \geq 0 \mid \text{there exists a } \delta\text{-matching between } A \text{ and } B\}$$

Lemma 2.3.11. *The bottleneck distance satisfies the triangle inequality: for any three subsets A, B and C of $\mathbb{R}^2$, we have*

$$d_B(A, C) \leq d_B(A, B) + d_B(B, C).$$

Proof. By Remark 2.3.9, if A and B are δ_1-matched and B and C are δ_2-matched, then A and C are $(\delta_1 + \delta_2)$-matched. The result follows by taking the infimum over all such δ_1 and δ_2. $\square$

Proposition 2.3.12. *Let $\langle p^*, q^* \rangle$ and $\langle r^*, s^* \rangle$ be intervals, and let $M = C\big(\langle p^*, q^* \rangle\big)$ and $N = C\big(\langle r^*, s^* \rangle\big)$ be the corresponding interval modules. Then*

$$d_I(M, N) \leq d^\infty((p, q), (r, s)).$$

Proof. We must show that for every $\delta > d^\infty((p,q),(r,s))$, M and N are δ-interleaved. We do this by noting that the maps $f : M \to N(\delta)$ and $g : N \to M(\delta)$ defined by

$$f_t = \begin{cases} \mathbb{1}_{\mathbf{k}} & \text{if } t \in \langle p^*, q^* \rangle \cap \langle r^* - \delta, s^* - \delta \rangle \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_t = \begin{cases} \mathbb{1}_{\mathbf{k}} & \text{if } t \in \langle p^* - \delta, q^* - \delta \rangle \cap \langle r^*, s^* \rangle \\ 0 & \text{otherwise,} \end{cases}$$

provide well-defined δ-inverse δ-interleavings. $\square$

Proposition 2.3.13. *Let $\langle p^*, q^* \rangle$ be an interval, and let $M = C\big(\langle p^*, q^* \rangle\big)$. Then*

$$d_I(M, 0) = \frac{1}{2}|q - p| = d^\infty((p, q), \Delta).$$

Proof. The only morphisms to and from the zero module are the zero maps. Therefore a pair of morphisms $f : M \to 0(\delta) = 0$ and $g : 0 \to M(\delta)$ are δ-inverse δ-interleavings if and only if $\varphi_M^{2\delta} = g(\delta) \circ f$, which simplifies to $\varphi_M^{2\delta} = 0$. This last equality holds if $2\delta > |q - p|$ and fails to hold if $2\delta < |q - p|$. $\square$

Theorem 2.3.14 (The Converse Stability Theorem for Decomposable Persistence Modules)**.** *Let M, N be decomposable persistence modules. Then*

$$d_I(M, N) \leq d_B(\mathbf{dgm}(M), \mathbf{dgm}(N)).$$

This theorem is called the *Converse Stability Theorem* because the reverse inequality, which was studied first, is true for pointwise finite-dimensional (PFD) persistence modules:

Theorem 2.3.15 (The Algebraic Stability Theorem for PFD Persistence Modules)**.** *Let M, N be PFD persistence modules. Then*

$$d_B(\mathbf{dgm}(M), \mathbf{dgm}(N)) \leq d_I(M, N).$$

We shall prove a stronger version of *Algebraic Stability Theorem* in Section 3 (see Theorem 3.5.2), following the proof originally appearing in [2] as Theorem 6.4.

Proof of the Converse Stability Theorem. Let $\delta \geq 0$, and suppose that there is a δ-matching σ of $\mathbf{dgm}(M)$ and $\mathbf{dgm}(N)$. Re-write M and N in the form

$$M = \bigoplus_{k \in \mathcal{K}} M_k$$
$$N = \bigoplus_{k \in \mathcal{K}} N_k$$

where we choose M_k, N_k such that either

- M_k is an I_k-submodule of M and N_k is a J_k-submodule of N, where I_k and J_k are a pair of matched intervals;
- M_k is an I_k-submodule of M where I_k is unmatched, and $N_k = 0$; or
- N_k is a J_k-submodule of N where J_k is unmatched, and $M_k = 0$.

By Propositions 2.3.12 and 2.3.13, $d_I(M_k, N_k) \leq \delta$ for each of the three possible cases above. By Proposition 2.2.11, $d_I(M, N) \leq \delta$. The result follows by taking the infimum over all $\delta \geq 0$ such that there exists a δ-matching of $\mathbf{dgm}(M)$ and $\mathbf{dgm}(N)$. □

3 Induced Matchings and the Algebraic Stability of Persistence Barcodes

In this section we summarize the results of Bauer and Lesnick from *Induced Matchings and the Algebraic Stability of Persistence Barcodes* [2], following their exposition.

3.1 Barcodes

Definition 3.1.1. A **barcode** is (a representation of) a multiset of intervals (see Definition 2.1.2).

Definition 3.1.2. Let M be a decomposable persistence module with

$$M \cong \bigoplus_{k \in \mathcal{K}} C(I_k).$$

The **barcode of M** is the multiset of intervals

$$\mathcal{B}_M = \{\!\{ I_k \mid k \in \mathcal{K} \}\!\}$$

Remark 3.1.3. The decorated persistence diagram and the barcode of a decomposable persistence module M contain exactly the same information: in fact, there is a canonical bijection of multisets $\mathbf{Dgm}(M) \to \mathcal{B}_M$ which sends $(p^*, q^*) \in \mathbf{Dgm}(M)$ to the interval $\langle p^*, q^* \rangle$.

Remark 3.1.4. By Theorem 2.1.4, the barcode of a decomposable persistence module does not depend on the choice of decomposition.

Remark 3.1.5. Recall that a persistence module M is pointwise finite-dimensional (PFD) if M_t is finite- dimensional for every $t \in \mathbb{R}$ (Definition 2.0.1), and that every PFD persistence module is decomposable (Theorem 2.1.7). In particular, *to every PFD persistence module M there corresponds a unique barcode $\mathcal{B}_M$.*

Remark 3.1.6. The barcode of $M(\delta)$ is simply the barcode of M shifted to the left by δ (see Definition 2.2.1 and Remark 2.2.2).

Given a barcode $\mathcal{D}$, we define a new barcode $\mathcal{D}^\epsilon$ by

$$\mathcal{D}^\epsilon = \{I \in \mathcal{D} \ : \ [t, t+\epsilon] \subset I \text{ for some } t \in \mathbb{R}\}.$$

In words, $\mathcal{D}^\epsilon$ is the collection of intervals in $\mathcal{D}$ of length strictly greater than ϵ together with the closed intervals of length ϵ.

Definition 3.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be barcodes. A $\boldsymbol{\delta}$**−matching** is a partial matching $\sigma : \mathcal{C} \nrightarrow \mathcal{D}$ such that

(i) $\mathcal{C}^{2\delta} \subset \mathrm{dom}(\sigma)$,

(ii) $\mathcal{D}^{2\delta} \subset \mathrm{im}(\sigma)$,

(iii) If $\sigma\big(\langle b,d\rangle\big) = \langle b',d'\rangle$ then

$$\begin{aligned}\langle b,d\rangle &\subset \langle b'-\delta, d'+\delta\rangle\\ \langle b',d'\rangle &\subset \langle b-\delta, d+\delta\rangle.\end{aligned}$$

The **bottleneck distance** $d_B(\cdot,\cdot)$ between barcodes $\mathcal{C}$ and $\mathcal{D}$ is the infimum over all $\delta \geq 0$ such that $\mathcal{C}$ and $\mathcal{D}$ are δ-matched:

$$d_B(\mathcal{C},\mathcal{D}) = \inf\{\delta \geq 0 \mid \mathcal{C} \text{ and } \mathcal{D} \text{ are } \delta\text{-matched}\}.$$

Remark 3.1.8. If M and N are persistence modules, then $d_B(\mathcal{B}_M, \mathcal{B}_N) = d_B(\mathbf{dgm}(M), \mathbf{dgm}(N))$, which justifies the reuse of the notation $d_B(\cdot,\cdot)$. Note however that a δ-matching of the barcodes $\mathcal{B}_M$ and $\mathcal{B}_N$ is strictly stronger than a δ-matching of persistence diagrams $\mathbf{dgm}(M)$ and $\mathbf{dgm}(N)$, in the sense that if $\mathcal{B}_M$ and $\mathcal{B}_N$ are δ-matched then $\mathbf{dgm}(M)$ and $\mathbf{dgm}(N)$ are δ-matched, but the converse is not true in general. Roughly speaking, this is because Definition 2.3.8 does not distinguish between distinct intervals with identical endpoints, as it "forgets" decorations.

For instance, let $M = C\big([p,q]\big)$ and $N = C\big((p+\delta, q-\delta)\big)$ where $p < q$ and $0 \leq 2\delta < q-p$. Then $\mathbf{dgm}(M) = \{(p,q)\} \subset \mathbb{R}^2$ and $\mathbf{dgm}(N) = \{(p+\delta, q-\delta)\} \subset \mathbb{R}^2$ are δ-matched (simply match the point (p,q) with $(p+\delta, q-\delta)$). However, $\mathcal{B}_M = \{[p,q]\}$ and $\mathcal{B}_N = \{(p+\delta, q-\delta)\}$ are not δ-matched: indeed, since $[p, p+2\delta] \subset [p,q]$, by condition (i) of Definition 3.1.7 $[p,q]$ must be matched to $(p+\delta, q-\delta)$ by any δ-matching; but since $[p,q] \not\subset (p,q)$, by condition (iii) $[p,q]$ and $(p+\delta, q-\delta)$ cannot be matched.

3.2 Dual Modules

Let $\mathbf{R}^{op} = (\mathbb{R}, \geq)$ be the opposite category of $\mathbf{R}$, that is the category with objects $t \in \mathbb{R}$ and a unique morphism $s \to t$ whenever $s \geq t$ (instead of whenever $s \leq t$). Let $\mathrm{Neg} : \mathbf{R} \to \mathbf{R}^{op}$ denote the functor which sends a real number t to $-t$ and the morphism $s \leq t$ to $-s \geq -t$. Let $(\cdot)^* : \mathbf{Vect} \to \mathbf{Vect}$ denote the duality (contravariant) functor. Given a persistence module $M : \mathbf{R} \to \mathbf{Vect}$, taking the duals of all vector spaces and transition maps gives a functor $M^\dagger : \mathbf{R}^{op} \to \mathbf{Vect}$. Define the **dual** of M to be the persistence module $M^* = M^\dagger \circ \mathrm{Neg} : \mathbf{R} \to \mathbf{Vect}$. Explicitly, M^* has vector spaces $(M^*)_t = (M_{-t})^*$ and transition maps $\varphi_{M^*}(s,t) = \varphi_M(-t,-s)^*$.

Given a morphism of persistence modules $f : M \to N$, we define the dual morphism $f^* : N^* \to M^*$ by letting $(f^*)_t = (f_{-t})^*$.

With these identifications, dualization is a contravariant endofunctor $(\cdot)^* : \mathbf{Vect}^{\mathbb{R}} \to \mathbf{Vect}^{\mathbb{R}}$. When M and N are PFD, under the canonical identifications $M = M^{**}$ and $N = N^{**}$ we have $f^{**} = f$.

For a barcode $\mathcal{D}$, let $\mathcal{D}^* = \{-I \ : \ I \in \mathcal{D}\}$ where we define $-I = \{-t \ : \ t \in I\}$.

Proposition 3.2.1. *If M is a PFD persistence module, then $\mathcal{B}_{M^*} = (\mathcal{B}_M)^*$.*

Proof. Without loss of generality, let $M = \bigoplus_{I \in \mathcal{B}_M} C(I)$. It suffices to show that $M^* \cong \bigoplus_{I \in \mathcal{B}_M} C(-I)$. We shall prove this in two steps:

(i) If $N = \bigoplus_{k \in \mathcal{K}} N_k$ is PFD then $N^* = \bigoplus_{k \in \mathcal{K}} N_k^*$.

(ii) For any interval $I \subset \mathbb{R}$, $C(I)^* \cong C(-I)$.

For (i), let $s \leq t$. Since the N_{-t} and N_{-s}, $-t \leq -s$ are finite dimensional, only finitely many of the terms $\varphi_{N_k}(-t,-s)$ in $\varphi_N(-t,-s) = \bigoplus_{k \in \mathcal{K}} \varphi_{N_k}(-t,-s)$ are nonzero. Therefore $\varphi_N(-t,-s)^* = \bigoplus_{k \in \mathcal{K}} \varphi_{N_k}(-t,-s)^*$, i.e. $\varphi_{N^*}(s,t) = \bigoplus_{k \in \mathcal{K}} \varphi_{N_k^*}(s,t)$ whenever $s \leq t$, which proves the result.

For (ii), observe that

$$(C(I)^*)_t = (C(I)_{-t})^* = \begin{cases} \mathbf{k}^*, & \text{if } t \in -I \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbb{1}_{\mathbf{k}}^* = \mathbb{1}_{\mathbf{k}^*}$, the transition maps of $C(I^*)$ are

$$\varphi_{C(I)^*}(s,t) = \varphi_{C(I)}(-t,-s)^* = \begin{cases} \mathbb{1}_{\mathbf{k}^*}, & \text{if } s,t \in -I \\ 0, & \text{otherwise.} \end{cases}$$

The isomorphism $\mathbf{k} \to \mathbf{k}^*$ sending $a \in \mathbf{k}$ to $a \cdot \mathbb{1}_{\mathbf{k}} \in \mathbf{k}^*$ therefore leads to an isomorphism $f_t : C(-I) \to C(I)^*$:

$$f_t(a) = \begin{cases} a \cdot \mathbb{1}_{\mathbf{k}}, & \text{if } t \in -I \\ 0, & \text{otherwise.} \end{cases}$$

Now applying (i) and (ii) above, we see that

$$M^* = \bigoplus_{I \in \mathcal{B}_M} C(I)^* \cong \bigoplus_{I \in \mathcal{B}_M} C(-I),$$

as required. $\square$

3.3 The Structure of Persistence Submodules and Quotients

A partially ordered set $(S, <)$ is said to be **enumerated** if there is an order-preserving injection $S \hookrightarrow \mathbb{N}$ into the positive integers with the standard order. We may write $S = \{s_1, s_2, ...\}$ with $s_1 < s_2 < ...$ where this sequence terminates if S is finite.

Given two enumerated sets $S = \{s_1, s_2, ...\}$ and $T = \{t_1, t_2, ...\}$ with $|S| \leq |T|$, there is a **canonical injection** $\alpha_S^T : S \hookrightarrow T$ defined by

$$\alpha_S^T(s_i) = t_i.$$

Remark 3.3.1. If S, T, U are enumerated sets with $|S| \leq |T| \leq |U|$, then the canonical injections trivially satisfy the composition law

$$\alpha_S^U = \alpha_T^U \circ \alpha_S^T.$$

Let M be a PFD persistence module and let $\mathbb{D}$ denote the decorated real numbers (see Definition 2.3.1). For any $b \in \mathbb{D}$, define $\langle b, \cdot \rangle_M$ to be (the representation of) the multiset of intervals $I \in \mathcal{B}_M$ of the form $I = \langle b, d \rangle$ for some $d \in \mathbb{D}$. Clearly

$$\mathcal{B}_M = \coprod_{b \in \mathbb{D}} \langle b, \cdot \rangle_M.$$

Order $\langle b, \cdot \rangle_M$ by reverse inclusion, so that larger intervals come before smaller ones. More precisely, let $\langle b, d \rangle_k$ denote the k^{th} interval of the form $\langle b, d \rangle$ in the representation of $\langle b, \cdot \rangle_M$. We say that $\langle b, d \rangle_k < \langle b, d' \rangle_{k'}$ if either $d > d'$, or $d = d'$ and $k < k'$.

Remark 3.3.2. $\left(\langle b, \cdot \rangle_M, < \right)$ is an enumerated set since $<$ defined above is clearly a total order and the number of predecessors of an interval $\langle b, d \rangle \in \left(\langle b, \cdot \rangle_M, < \right)$ must be finite for M to be PFD.

Dually, define $\langle \cdot, d \rangle_M$ to be the collection of intervals $I \in \mathcal{B}_M$ of the form $\langle b, d \rangle$ for some $b \in \mathbb{D}$, so that

$$\mathcal{B}_M = \coprod_{d \in \mathbb{D}} \langle \cdot, d \rangle_M.$$

Ordering distinct intervals by reverse inclusion (and ordering repeated intervals as above) makes $\langle \cdot, d \rangle_M$ into an enumerated set. Note that in this case $\langle b, d \rangle < \langle b', d \rangle$ if $b < b'$.

Definition 3.3.3. A morphism $j : M \to N$ of persistence modules M and N is a **monomorphism** if $j_t : M_t \to N_t$ is *injective* for all $t \in \mathbb{R}$.

A morphism $q : M \to N$ of persistence modules M and N is an **epimorphism** if $q_t : M_t \to N_t$ is *surjective* for all $t \in \mathbb{R}$.

We shall use the notation $j : M \hookrightarrow N$ for monomorphisms and $q : M \twoheadrightarrow N$ for epimorphisms. The reader may verify that this definition agrees with the standard categorical definitions of monomorphisms and epimorphisms in the category $\mathbf{Vect}^{\mathbf{R}}$ of persistence modules.

Theorem 3.3.4 (Structure of Persistence Submodules and Quotients)**.** *Let M and N be PFD persistence modules.*

(i) If there is a monomorphism $M \hookrightarrow N$, then for each $d \in \mathbb{D}$

$$\left|\left\langle \cdot, d\right\rangle_M\right| \leq \left|\left\langle \cdot, d\right\rangle_N\right|$$

and the union of the canonical injections $\left\langle \cdot, d\right\rangle_M \hookrightarrow \left\langle \cdot, d\right\rangle_N$, $d \in \mathbb{D}$ defines an injection $\mathcal{B}_M \hookrightarrow \mathcal{B}_N$ sending each interval $\left\langle b, d\right\rangle \in \mathcal{B}_M$ to an interval $\left\langle b', d\right\rangle$ with $b' \leq b$ (equivalently, with $\left\langle b', d\right\rangle \leq \left\langle b, d\right\rangle$).

(ii) Dually, if there is epimorphism $M \twoheadrightarrow N$, then for each $b \in \mathbb{D}$

$$\left|\left\langle b, \cdot\right\rangle_M\right| \geq \left|\left\langle b, \cdot\right\rangle_N\right|$$

and the union of the canonical injections $\left\langle b, \cdot\right\rangle_M \hookleftarrow \left\langle b, \cdot\right\rangle_N$, $b \in \mathbb{D}$ defines an injection $\mathcal{B}_M \hookleftarrow B_N$ sending each $\left\langle b, d\right\rangle \in \mathcal{B}_N$ to an interval $\left\langle b, d'\right\rangle$ with $d \leq d'$ (equivalently, with $\left\langle b, d\right\rangle \geq \left\langle b, d'\right\rangle$).

Remark 3.3.5. Abusing notation, we shall refer to the injection in part (i) (resp. (ii)) of Theorem 3.3.4 as the **canonical injection** $\mathcal{B}_M \hookrightarrow \mathcal{B}_N$ (resp. $\mathcal{B}_M \hookleftarrow \mathcal{B}_N$) when the conditions of (i) (resp. (ii)) hold.

Informally, the canonical injection in Theorem 3.3.4(i) (resp. (ii)) sends an interval in M (resp. N) to a larger interval in N (resp. M) with the same right (resp. left) endpoint. Here "larger" means larger in the sense of length, not in the sense of the partial order we have defined on intervals – for instance, the interval $[0, 2]$ is "larger" than the interval $[0, 1]$, but $[0, 2] < [0, 1]$ in the partial order defined above. Note that these injections are canonical with respect to the representation of $\mathcal{B}_M$ and $\mathcal{B}_N$, not the multisets $\mathcal{B}_M$ and $\mathcal{B}_N$, since permuting identical intervals in $\mathcal{B}_M$ without a predefined order would yield an equally sensible injection.

For $I = \langle b, d \rangle$, define $\langle \cdot, I \rangle_M$ to be the multiset

$$\langle \cdot, I \rangle_M := \{J = \langle b', d \rangle_k \in \langle \cdot, d \rangle_M \mid \langle b', d \rangle \ \leq I\},$$

the multiset of predecessors of I in $\langle \cdot, d \rangle_M$ with respect to reverse inclusion.

Lemma 3.3.6. *Let I be an interval. If there exists a monomorphism of PFD persistence modules $j : M \hookrightarrow N$ then*

$$|\langle \cdot, I \rangle_M| \leq |\langle \cdot, I \rangle_N|.$$

Proof. Let $I = \langle b, d \rangle$. Without loss of generality, assume

$$M = \bigoplus_{J \in \mathcal{B}_M} C(J)$$
$$N = \bigoplus_{J \in \mathcal{B}_N} C(J).$$

Let $U \subset M$ and $V \subset N$ be the submodules

$$U = \bigoplus_{J \in \langle \cdot, I \rangle_M} C(J)$$
$$V = \bigoplus_{J \in \langle \cdot, I \rangle_N} C(J).$$

Given an interval J, we say that $t > J$ if $t > s$ for every $s \in J$. Since M and N are PFD, for any $s \in I$ there must be finitely many intervals $J \in \mathcal{B}_M \cup \mathcal{B}_N$ with $s \in J$; it follows that there must be some $t \in I$ such that $t > \langle b', d' \rangle$ whenever $\langle b', d' \rangle \in \mathcal{B}_M \cup \mathcal{B}_N$ with $b' \leq b$ and $d' < d$. Clearly $\dim U_t = |\langle \cdot, I \rangle_M|$ and $\dim V_t = |\langle \cdot, I \rangle_N|$.

We claim that $j_t(U_t) \subset V_t$. By the choice of t, we have

$$U_t = \bigcap_{\substack{s \in I \\ s \leq t}} \operatorname{im} \varphi_M(s, t) \ \cap \ \bigcap_{r > I} \ker \varphi_M(t, r)$$
$$V_t = \bigcap_{\substack{s \in I \\ s \leq t}} \operatorname{im} \varphi_N(s, t) \ \cap \ \bigcap_{r > I} \ker \varphi_N(t, r).$$

For each $s \in I$, we have $j_t(\operatorname{im}\varphi_M(s,t)) \subset \operatorname{im}\varphi_N(s,t)$ by the commutativity of the diagram

$$\begin{tikzcd} M_s \arrow[r, "\varphi_M(s,t)"] \arrow[d, "j_s"] & M_t \arrow[d, "j_t"] \\ N_s \arrow[r, "\varphi_N(s,t)"] & N_t \end{tikzcd}$$

Similarly, $j_t(\ker\, \varphi_M(t,r)) \subset \ker\, \varphi_N(t,r)$ by the commutativity of the diagram

$$\begin{tikzcd} M_t \arrow[r, "\varphi_M(t,r)"] \arrow[d, "j_t"] & M_t \arrow[d, "j_r"] \\ N_t \arrow[r, "\varphi_N(t,r)"] & N_r \end{tikzcd}$$

We conclude that $j_t(U_t) \subset V_t$. Since j_t is an injection, we have $\dim U_t \leq \dim V_t$, completing the proof. $\square$

Proof of Theorem 3.3.4. Suppose that there exists a monomorphism $M \hookrightarrow N$. To show that $\left|\left\langle \cdot, d\right\rangle_M\right| \leq \left|\left\langle \cdot, d\right\rangle_N\right|$, it suffices to show that if $i \leq \left|\left\langle \cdot, d\right\rangle_M\right|$ then $i \leq \left|\left\langle \cdot, d\right\rangle_N\right|$. Let $I = \left\langle b, d\right\rangle$ denote the i^{th} interval of $\left\langle \cdot, d\right\rangle_M$. Then for $1 \leq j \leq i$, $\left\langle \cdot, I\right\rangle_M$ contains the j^{th} interval of $\left\langle \cdot, d\right\rangle_M$ and so

$$i \leq \left|\left\langle \cdot, I\right\rangle_M\right| \leq \left|\left\langle \cdot, I\right\rangle_N\right| \leq \left|\left\langle \cdot, d\right\rangle_N\right|$$

where the second inequality follows from Lemma 3.3.6.

Let $I' = \left\langle b', d\right\rangle$ denote the i^{th} interval of $\left\langle \cdot, d\right\rangle_N$. Then the canonical injection $\mathcal{B}_M \hookrightarrow \mathcal{B}_N$ sends I to I'. Since $i \leq \left|\left\langle \cdot, I\right\rangle_N\right|$, we must have $I' \in \left\langle \cdot, I\right\rangle_N$ or equivalently $b' \leq b$. This completes the proof of part (i).

Part (ii) follows from part (i) by a duality argument. Given an epimorphism of PFD persistence modules $q : M \twoheadrightarrow N$, the dual $q^* : N^* \hookrightarrow M^*$ is a monomorphism. By part (i), q^* induces an injection $\iota : \mathcal{B}_{N^*} \hookrightarrow \mathcal{B}_{M^*}$. By Proposition 3.2.1, this in turn induces an injection $\mathcal{B}_N \hookrightarrow \mathcal{B}_M$ by sending $I \in \mathcal{B}_N$ to $-\iota(-I)$, which is exactly the canonical injection. By part (i), we see that $I = \left\langle b, d\right\rangle$ gets sent to $I' = -\left\langle -d', -b\right\rangle = \left\langle b, d'\right\rangle$ with $-d' \leq -d$, proving part (ii). $\square$

3.4 Induced Matchings of Barcodes

Given a morphism of PFD persistence modules $f : M \to N$, we shall define a partial matching $\chi_f : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ induced by f.

We first define this partial matching for monomorphisms and epimorphisms. If $j : M \hookrightarrow N$ is a monomorphism, then we define $\chi_j : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ to be the canonical injection $\iota : \mathcal{B}_M \hookrightarrow \mathcal{B}_N$ of Theorem 3.3.4(i). If $q : M \twoheadrightarrow N$ is an epimorphism, then we define $\chi_q : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ to be the inverse of the canonical injection $\iota' : \mathcal{B}_N \hookrightarrow \mathcal{B}_M$ of Theorem 3.3.4(ii) (namely the unique matching with domain $\operatorname{im} \iota'$ such that $\chi_q \circ \iota' = \mathbb{1}_{\mathcal{B}_N}$).

In general, if $f : M \to N$ is a morphism of PFD persistence modules, then f factors canonically as

$$M \xrightarrow{q_f} \operatorname{im}(f) \xhookrightarrow{j_f} N.$$

We define the partial matching $\chi_f : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ to be the composition $\chi_f = \chi_{j_f} \circ \chi_{q_f}$.

Proposition 3.4.1. *Let $f : M \to N$ be a morphism of PFD persistence modules. Suppose $\chi_f(\langle b, d\rangle) = \langle b', d'\rangle$. Then*

$$b' \leq b < d' \leq d.$$

In words, induced matchings shift the endpoints of an interval to the left.

Proof. By Theorem 3.3.4, we have $\chi_{q_f}(\langle b, d\rangle) = \langle b, d'\rangle$ with $d' \leq d$ and $\chi_{j_f}(\langle b, d'\rangle) = \langle b', d'\rangle$ with $b' \leq b$. The middle inequality $b < d'$ holds because $\langle b, d'\rangle$ is an interval. □

Proposition 3.4.2. *χ is functorial when restricted to the subcategory of monomorphisms of PFD persistence modules. Dually, χ is functorial when restricted to the subcategory of epimorphisms of PFD persistence modules.*

Proof. Let $j_1 : M \to N$ and $j_2 : N \to P$ be monomorphisms of PFD persistence modules. By definition, $\chi_{j_1} : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ is the disjoint union of the canonical injections $\langle \cdot, d\rangle_M \hookrightarrow \langle \cdot, d\rangle_N$, and similarly for $\chi_{j_2} : \mathcal{B}_N \nrightarrow \mathcal{B}_P$ and $\chi_{j_2 \circ j_1} : \mathcal{B}_M \nrightarrow \mathcal{B}_P$. By Remark 3.3.1, it follows that $\chi_{j_2 \circ j_1} = \chi_{j_2} \circ \chi_{j_1}$. Thus χ is functorial when restricted to the subcategory of monomorphisms in $\mathbf{vect}^{\mathbb{R}}$.

The result for epimorphisms follows by essentially the same argument, together with the fact that the operation of reversing partial matchings is functorial. Thus χ is functorial when restricted to the subcategory of epimorphisms in $\mathbf{vect}^{\mathbb{R}}$. □

For N a persistence module and $\epsilon > 0$, we define a submodule N^ϵ of N by setting

$$N_t^\epsilon = \operatorname{im} \varphi_N(t-\epsilon, t)$$

for all $t \in \mathbb{R}$, with transition maps the restriction of φ_N to N^ϵ.

Definition 3.4.3. A persistence module N is said to be $\boldsymbol{\epsilon}$**−trivial** if the ϵ-transition morphism $\varphi_N^\epsilon : N \to N(\epsilon)$ is the zero morphism, or equivalently if $N^\epsilon = 0$.

Theorem 3.4.4 (Induced Matching Theorem)**.** *Let $f : M \to N$ be a morphism of PFD persistence modules, and suppose that $\chi_f\big(\langle b, d\rangle\big) = \langle b', d'\rangle$. Then*

(i) $b' \leq b < d' \leq d$.

(ii) If coker$(f) = N/$im(f) is ϵ-trivial, then $\mathcal{B}_N^\epsilon \subset$ im(χ_f) and

$$b' \leq b \leq b' + \epsilon.$$

(iii) Dually, if ker(f) is ϵ-trivial, then $\mathcal{B}_M^\epsilon \subset$ dom(χ_f) and

$$d - \epsilon \leq d' \leq d.$$

Proof. Part (i) is exactly Proposition 3.4.4.

Let N^ϵ be as defined immediately before Definition 3.4.3, and recall that $\mathcal{B}_N^\epsilon$ is the collection of intervals in $\mathcal{B}_N$ containing a closed interval of length ϵ. First observe that to obtain $\mathcal{B}_{N^\epsilon}$, one simply shifts the left endpoints of intervals in $\mathcal{B}_N^\epsilon$ to the right by ϵ:

$$\mathcal{B}_{N^\epsilon} = \{\langle b+\epsilon, d\rangle \mid \langle b, d\rangle \in \mathcal{B}_N^\epsilon\}.$$

Next, let $j_\epsilon : N^\epsilon \hookrightarrow N$ be the inclusion map. By the definition of χ_{j_ϵ}, we see that $\chi_{j_\epsilon}\big(\langle b+\epsilon, d\rangle\big) = \langle b, d\rangle$ for every $\langle b, d\rangle \in \mathcal{B}_N^\epsilon$ and consequently that $\operatorname{im}(\chi_{j_\epsilon}) = \mathcal{B}_N^\epsilon$.

To prove Part (ii), suppose that coker(f) is ϵ-trivial. Then $N^\epsilon \subset$ im(f). (Indeed, this follows because $\varphi_{\operatorname{coker}(f)}(t-\epsilon, t) = 0$ if and only if $\operatorname{im} \varphi_N(t-\epsilon, t) \subset \operatorname{im}(f_t)$.) This implies that the following diagram commutes, where each map is the inclusion:

$$\begin{tikzcd}
 & N \\
N^\epsilon \arrow[ur, hook, "j_\epsilon"] \arrow[dr, hook, "j"'] & \\
 & \operatorname{im}(f) \arrow[uu, hook, "j_f"']
\end{tikzcd}$$

By Proposition 3.4.2, $\chi_{j_\epsilon} = \chi_{j_f} \circ \chi_j$. Furthermore, $\chi_f = \chi_{j_f} \circ \chi_{q_f}$ by the definition of χ_f. Hence the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{B}_N & \\
\mathcal{B}_{N^\epsilon} \arrow[ur, hook, "\chi_{j_\epsilon}"] \arrow[dr, hook, "\chi_j"'] & & \mathcal{B}_M \arrow[ul, "\chi_f"'] \arrow[dl, "\chi_{q_f}"] \\
& \mathcal{B}_{\mathrm{im}(f)} \arrow[uu, hook, "\chi_{j_f}"] &
\end{tikzcd}$$

By the commutativity of the left triangle, $\mathcal{B}_N^\epsilon = \mathrm{im}(\chi_{j_\epsilon}) \subset \mathrm{im}(\chi_{j_f})$. By the definition of the induced partial matchings, $\mathrm{im}(\chi_{j_f}) = \mathrm{im}(\chi_f)$ and so

$$\mathcal{B}_N^\epsilon \subset \mathrm{im}(\chi_f),$$

as claimed.

To finish the proof of Part (ii), we must show that whenever $\chi_f\big(\langle b, d\rangle\big) = \langle b', d'\rangle$ the inequality

$$b' \leq b \leq b' + \epsilon$$

holds. The lefthand inequality follows from Part (i), and the righthand inequality follows by the commutativity of the left triangle as follows. By the definition of the induced partial matching, we have $\chi_{j_f}\big(\langle b, d'\rangle\big) = \langle b', d'\rangle$. Now $\chi_{j_\epsilon}\big(\langle b' + \epsilon, d'\rangle\big) = \langle b', d'\rangle$, so by the commutativity of the left triangle we have $\chi_j\big(\langle b' + \epsilon, d'\rangle\big) = \langle b, d'\rangle$. The inequality $b \leq b' + \epsilon$ therefore follows by applying Theorem 3.3.4 to the monomorphism $j : N^\epsilon \hookrightarrow N$.

Finally, the proof of Part (ii) dualizes readily to a proof of Part (iii). □

3.5 An Explicit Formulation of the Algebraic Stability Theorem

We shall see that the Algebraic Stability Theorem (Theorem 2.3.15 above and the stronger Theorem 3.5.2 below) follows readily from the Induced Matching Theorem (Theorem 3.4.4).

Lemma 3.5.1. *If $f : M \to N(\delta)$ is a δ-interleaving morphism, then both ker(f) and coker(f) are 2δ-trivial.*

Proof. By the definition of a δ-interleaving, there exists a morphism $g : N \to M(\delta)$ such that

$$g(\delta) \circ f = \varphi_M^{2\delta}$$

and

$$f(\delta) \circ g = \varphi_N^{2\delta}.$$

The first equality implies that $\ker(f)$ is 2δ-trivial, and the second equality implies that $\operatorname{coker}(f)$ is 2δ-trivial. □

For a persistence module N and $\delta \geq 0$, there is a bijection $r_\delta : \mathcal{B}_{N(\delta)} \to \mathcal{B}_N$ given by $r_\delta\big(\langle b, d\rangle\big) = \langle b+\delta, d+\delta\rangle$ for each interval $\langle b, d\rangle \in \mathcal{B}_{N(\delta)}$.

Theorem 3.5.2 (Explicit Formulation of the Algebraic Stability Theorem)**.** *If $f : M \to N(\delta)$ is a δ-interleaving of PFD persistence modules, then $r_\delta \circ \chi_f : \mathcal{B}_M \nrightarrow \mathcal{B}_N$ is a δ-matching. In particular,*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) \leq d_I(M, N).$$

Proof. By Lemma 3.5.1, $\ker(f)$ and $\operatorname{coker}(f)$ are both 2δ-trivial. By the Induced Matching Theorem, we have that $\mathcal{B}_M^{2\delta} \subset \operatorname{dom}(\chi_f)$ and $\mathcal{B}_N^{2\delta} \subset \operatorname{im}(\chi_f)$. If $\chi_f\big(\langle b, d\rangle\big) = \langle b', d'\rangle$, we have that

$$b' \leq b \leq b' + 2\delta \ \text{ and } \ d - 2\delta \leq d' \leq d,$$

or equivalently, in terms of the endpoints of $r_\delta \circ \chi_f\big(\langle b, d\rangle\big) = \langle b'+\delta, d'+\delta\rangle$,

$$(b' + \delta) - \delta \leq b \leq (b' + \delta) + \delta \ \text{ and } \ d - \delta \leq (d' + \delta) \leq d + \delta.$$

This verifies that $r_\delta \circ \chi_f$ is indeed a δ-matching. □

Combining Theorem 2.3.14 (The Converse Stability Theorem for Decomposable Persistence Modules) and 2.1.7 (every PFD persistence module is decomposable), recalling that $d_B(\mathcal{B}_M, \mathcal{B}_N) = d_B(\mathbf{dgm}(M), \mathbf{dgm}(N))$ for persistence modules M and N, we see that the inequality of Theorem 3.5.2 is actually an equality:

Theorem 3.5.3 (The Isometry Theorem for PFD Persistence Modules)**.** *If M and N are PFD persistence modules, then*

$$d_B(\mathcal{B}_M, \mathcal{B}_N) = d_I(M, N).$$

This result concludes our study of persistence modules.

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