Moduli spaces and modular operads

Jeffrey Giansiracusa [1]

Abstract

We describe a generalised ribbon graph decomposition for a broad class of moduli spaces of geometric structures on surfaces (with nonempty boundary), including moduli of spin surfaces, r-spin surfaces, surfaces with a principle G-bundle, surfaces with maps to a background space, surfaces with Higgs bundle, etc.

1 Introduction

This paper is an expansion of some ideas that I first talked about in 2012 in the MIMS conference on Operads and Configuration Spaces. Here I shall give a more detailed account, though still not a complete one, of a certain theorem about modular envelopes. The full details will appear in a future paper; in this note I will try to be expository and focus on illuminating the central ideas without being overly concerned by technical details that might otherwise obscure some of the conceptual clarity of the arguments.

Fix a class ψ of geometric structures on surfaces. For example, one could take orientations, principal G-bundles, or spin structures, etc. Associated to any surface Σ is the *space* $\psi(\Sigma)$ of all such structures on that surface. Taking the homotopy quotient by the diffeomorphism group yields a homotopy theoretic moduli space of surfaces with ψ-structure. If we consider surfaces with some marked intervals along the boundary, and ψ-structures that have a fixed value on each marked interval, then we can glue the intervals together and the result is a modular operad.

denoted $\mathcal{M}_\psi$. (If, instead of a single fixed value on the intervals, we allow one of several fixed valued then the result is instead a coloured modular operad). These moduli spaces are the objects we wish to study. The idea of this work is to decompose them, in a sense, into moduli spaces of discs with ψ-structure. The modular operad $\mathcal{M}_\psi$ contains a sub-cyclic operad $\mathcal{D}_\psi$ of moduli spaces of discs with ψ-structure. Our main result is that $\mathcal{M}_\psi$ is freely generated (in a homotopical sense) as a modular operad over this sub-cyclic operad. I.e., the derived modular envelope of $\mathcal{D}_\psi$ is weakly equivalent to $\mathcal{M}_\psi$.

This result was inspired by the work of Costello. As part of his groundbreaking work in the homotopy theory of open-closed topological field theories [9], he gave a new perspective on the very important idea of describing the moduli space of Riemann surfaces with ribbon graphs in [8, 10]. He proved that the derived (i.e., homotopy invariant) modular envelope of the associative operad gives a model for the modular operad of moduli spaces of Riemann surfaces with open-string type gluing for the compositions. A point in this modular envelope can be described as a graph equipped with lengths on all of its edges and a cyclic order of the edges incident at each vertex — i.e., a metric ribbon graph. Thus the moduli space of ribbon graphs is equivalent to the moduli space of Riemann surfaces.

Costello's proof used geometry and analysis on a certain partial compactification of the moduli space of Riemann surfaces. Thus it appears his argument is not suited to more homotopy theoretic contexts such as the one considered in this paper. In [13], I gave a different proof of Costello's modular envelope theorem. This proof instead rested on the well-known contractibility of the arc complex of a surface. This new argument lead to an adaptation to dimension 3: the derived modular envelope of the framed little 2-discs is equivalent to the modular operad of moduli spaces of 3-dimensional handlebodies.

Here we instead focus of refining and generalising the argument of [13] in dimension 2. When the structures being considered are principal G-bundles then we expect this result will lead to a G-equivariant version of Costello's open-closed TFT theorem.

2   Operads

A *cyclic operad* in $\mathscr{C}$ is a functor $\mathcal{P}$ from the category of finite sets and bijections to $\mathscr{C}$ together with composition maps

$$\mathcal{P}(I) \otimes P(J) \overset{i \circ j}{\to} \mathcal{P}(I \sqcup J \smallsetminus \{i, j\}),$$

for $i \in I$ and $j \in J$, satisfying an associativity condition and natural in (I, i) and (J, j). One can think of $\mathcal{P}$ as a collection of abstract "electrical circuit components," where $\mathcal{P}(I)$ as a set/space of components with terminals given by the set I. The composition maps correspond to wiring terminals together to produce new components; terminals can only be glued in pairs (no trivalent connections) and in a cyclic operad two components can only be glued together in at most one place. Allowing multiple gluings leads to the following definition.

A *modular operad* in $\mathscr{C}$ is a cyclic operad $\mathcal{Q}$ together with natural self-composition maps $\mathcal{Q}(I) \overset{\circ_{i,j}}{\to} \mathcal{Q}(I \smallsetminus \{i, j\})$ that commute with the cyclic operad composition maps and with each other.

Example 2.1. 1. The commutative modular operad is the constant functor sending each finite set to a point.

2. The associative cyclic operad $\mathcal{A}ssoc$ sends I to the set of cyclic orders on I.

We will need a slight generalisation in which there are different types of terminals and two terminals can only be connected if they are of the same type. The types are called *colours*. Fix a set Λ, which we will call the set of colours. A Λ-coloured set I consists of a finite set with a map to Λ. A morphism $I \to I'$ of coloured sets is a bijection that respects the colours. A *coloured cyclic operad* $\mathcal{P}$ is a functor from the category of coloured sets to $\mathscr{C}$ together with a collection of composition maps $_i\circ_j$ as before, but now only defined when i and j have the same colour. A *coloured modular operad* is defined analogously, where the self-composition maps $\circ_{ij}$ also only defined when i and j have the same colour.

2.1 Homotopy theory of cyclic and modular operads

Berger and Batanin [7] have recently constructed fully satisfactory Quillen model category structures on cyclic and modular operads. When talking about derived constructions such as the derived modular envelope, one could work with the model category structures. However, we take a more pragmatic approach, since the modular envelope is the only functor we ever have to derive, and our construction of the derived functor will be manifestly homotopy invariant due to the homotopy invariance of

homotopy colimits. We need only the following definition. A morphism $\mathcal{P} \to \mathcal{P}'$ of cyclic or modular operads in spaces is a weak equivalence if each map of spaces $\mathcal{P}(I) \to \mathcal{P}'(I)$ is a weak equivalence.

3   Category theory and homotopy theory

3.1   The nerve of a category

Let $\mathcal{C}$ be a category, which we assume is *small*, meaning that the objects form a set rather than a class. (E.g., The category of all sets is not small, but the category of all subsets of a fixed big set is small.) We can associate a simplicial set (and hence a space) with $\mathcal{C}$ via the *nerve* construction.

The nerve of $\mathcal{C}$, denoted $N(\mathcal{C})$ is the simplicial set whose 0-simplices are the objects of $\mathcal{C}$, 1-simplices are the morphisms of $\mathcal{C}$, 2-simplices are the 2-simplex shaped diagrams in $\mathcal{C}$

$$
\begin{array}{ccc}
X_0 & & \\
\downarrow f & \searrow^{g \circ f} & \\
X_1 & \xrightarrow{g} & X_2,
\end{array}
$$

and so on. In general, the n-simplices $N(\mathcal{C})_n$ are the set of composable n-tuples of morphisms,

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n.$$

We will write $B\mathcal{C}$ for the geometric realisation of the nerve.

It is easy to see that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of nerves $N_\bullet\mathcal{C} \to N_\bullet\mathcal{D}$. A natural transformation $F \to F'$ induces a homotopy between the corresponding maps. From this it follows that an adjoint pair (F, G) induces a homotopy equivalence of nerves.

If a category $\mathcal{C}$ has an initial object u then there is a natural transformation form the constant functor with value u to the identity, and so the nerve of $\mathcal{C}$ is contractible. Likewise, existence of a final object implies contractibility of the nerve.

3.2   The fundamental group of the nerve of a category

While computing the higher homotopy groups of a space is usually very difficult, there is a convenient recipe for computing the fundamental group of the nerve of a category.

Given $\mathscr{C}$, let $\mathscr{C}[\mathscr{C}^{-1}]$ denote the category formed by adjoining inverses to all of the arrows in $\mathscr{C}$ (see [6, §1.1]) This localisation is clearly a *groupoid* (all morphisms are invertible). If two objects of a groupoid lie in the same connected component then their automorphism groups are isomorphic (by an isomorphism that is unique up to conjugation). Given an object x of $\mathscr{C}$, the fundamental group of $N\mathscr{C}$ based at x is canonically isomorphic to the automorphism group of x in the groupoid $\mathscr{C}[\mathscr{C}^{-1}]$.

3.3   Strict 2-categories

A *strict 2-category* is a category enriched in $\mathscr{C}at$. I.e., it consists of a class of objects $\mathrm{Obj}\mathscr{C}$, a category $\mathrm{Hom}_{\mathscr{C}}(a, b)$ for each pair of objects, and composition functors

$$\mathrm{Hom}_{\mathscr{C}}(a, b) \times \mathrm{Hom}_{\mathscr{C}}(b, c) \to \mathrm{Hom}_{\mathscr{C}}(a, c)$$

that are strictly associative and for which a unit exists in $\mathrm{Hom}_{\mathscr{C}}(a, a)$. The objects of the hom categories are called *1-morphisms* and the morphisms of the hom categories are called *2-morphisms*.

Example 3.1. Let $\mathscr{T}op^2$ denote the strict 2-category whose objects are spaces, and for which $\mathrm{Hom}(X, Y)$ is the groupoid of maps and homotopy classes of homotopies. For example, the groupoid of morphisms from a point to a circle is equivalent to the group $\mathbb{Z}$ (i.e., one object and automorphism group $\mathbb{Z}$).

Example 3.2. Let $\mathscr{C}at^2$ denote the strict 2-category whose objects are small categories and whose hom categories are the categories of functors and natural transformations.

Example 3.3. Since a set can be considered as a category with no non-identity morphism, an ordinary category can be considered as a 2-category in which there are no non-identity 2-morphisms.

Remark 3.4. In this paper, all strict 2-categories that arise will have the property that all 2-morphisms are in fact isomorphisms. Such a 2-category is sometimes called a (2,1)-category. Strict 2-categories are a restricted class of 2-categories. More generally, one often wants to work with weak or lax 2-categories, where the associativity and unit conditions only hold up to natural transformations (which must then satisfy some conditions). We will have no need of these more sophisticated notions in this paper.

A *strict 2-functor* between strict 2-categories $F : \mathscr{C} \to \mathscr{D}$ is a map $F : \mathrm{Obj}\mathscr{C} \to \mathscr{D}$ together with a functor $\mathrm{Hom}(a, b) \to \mathrm{Hom}(F(a), F(b))$ for each pair of objects such that these functors are strictly compatible with the composition functors. We will usually abbreviate and call this a functor.

We will leave it as an exercise to spell out precisely what a natural transformation between strict 2-functors is.

A strict 2-category $\mathscr{C}$ has a nerve $N(\mathscr{C})$ that is a bisimplicial set. It is constructed as follows. First one replaces all the hom categories with their nerves to obtain a simplicial category. Then the nerve of this simplicial category yields a bisimplicial set that is the nerve of $\mathscr{C}$. As in the case of 1-categories, a strict 2-functor induces a map of nerves, and a natural transformation induces a homotopy. In particular, observe that if $F : \mathscr{C} \to \mathscr{C}at^2$ is a strict 2-functor then taking the realisation of the nerve pointwise yields a strict 2-functor $\mathscr{C} \to \mathscr{T}op^2$.

3.4 Over categories and Quillen's Theorems A and B

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor. Given an object $b \in \mathscr{B}$, one can define a category of objects in $\mathscr{A}$ over b. This is denoted $F \downarrow b$ (or $\mathscr{B} \downarrow b$ when F is the identity functor) and is called the *over category of F based at b* (some people instead call it the comma category). Its objects are pairs consisting of an object $a \in \mathscr{A}$ and a morphism $g : F(a) \to b$ in $\mathscr{B}$. A morphism from $F(a) \xrightarrow{g} b$ to $F(a') \xrightarrow{g'} b$ consists of a morphism $h : a \to a'$ in $\mathscr{A}$ such that the diagram

$$
\begin{array}{ccc}
F(a) & \xrightarrow{\ F(h)\ } & F(a') \\
& \searrow_{g} \quad \swarrow_{g'} & \\
& b &
\end{array}
$$

in $\mathscr{B}$ commutes. Observe that a morphism $f : b \to b'$ in $\mathscr{B}$ induces a functor $f_* : (F \downarrow b) \to (F \downarrow b')$. There is also a canonical projection functor $(F \downarrow b) \to \mathscr{A}$ given by forgetting the morphism to b.

Over categories can be thought of as a category-theoretic analogue of homotopy fibres. In fact, Quillen's Theorems A and B are instances of this analogy. If the homotopy fibre of a map is contractible then the map is a weak equivalence.

Theorem 3.1 (Quillen's Theorem A). *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor and suppose that for each object b of $\mathscr{B}$ the nerve of the over category $F \downarrow b$ is contractible. Then F induces a weak equivalence of nerves.*

This is a special case of a more general theorem that allows one to identify the homotopy fibre of a map of nerves induced by a functor.

Theorem 3.2 (Quillen's Theorem B)**.** *Let $F : \mathscr{A} \to \mathscr{B}$ be a functor and suppose for each morphism $f : b \to b'$ in $\mathscr{B}$ the corresponding functor $f_* : (F \downarrow b) \to (F \downarrow b')$ induces a weak equivalence of nerves. Then $(F \downarrow b) \to \mathscr{A} \to \mathscr{B}$ is a homotopy fibre sequence.*

The construction of over categories and Quillen's Theorems A and B have extensions to strict 2-categories. See [1] and [2] for details. Given a strict 2-functor $F : \mathcal{A} \to \mathcal{B}$ and an object $x \in \text{Obj}\mathcal{B}$, there is an over 2-category $(F \downarrow x)$. It objects are pairs $(a \in \text{Obj}\mathcal{A}, f : F(a) \to x)$. A morphism from $F(a_1) \xrightarrow{f_1} x$ to $F(a_2) \xrightarrow{f_2} x$ consists of a morphism $g : a_1 \to a_2$ in $\mathcal{A}$ and a 2-morphism in $\mathcal{B}$ from f_1 to $f_2 \circ F(g)$. A 1-morphism $x \to x'$ in $\mathcal{B}$ induces a strict *translation* 2-functor $(F \downarrow x) \to (F \downarrow x')$.

Theorem 3.3 (Theorem B for 2-categories,[2])**.** *If all of the translation functors induce homotopy equivalences then*

$$N(F \downarrow x) \to N(\mathcal{A}) \to N(\mathcal{B})$$

is a homotopy fibre sequence for any object x.

3.5   Left Kan extension

Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be categories. Given a functor $f : \mathscr{A} \to \mathscr{B}$, precomposition with f sends a functor $\mathscr{B} \to \mathscr{C}$ to a functor $\mathscr{A} \to \mathscr{C}$. This defines a functor

$$f^* : Fun(\mathscr{B}, \mathscr{C}) \to Fun(\mathscr{A}, \mathscr{C}).$$

It turns out that this f^* admits a left adjoint $f_!$, which is called the *left Kan extension.*

3.6   Left Kan extensions and homotopy left Kan extensions

Let $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ be categories with $\mathscr{C}$ cocomplete. Consider functors

$$\mathscr{A} \xrightarrow{F} \mathscr{C}$$
$$G \downarrow $$
$$\mathscr{B}.$$

Recall that the **left Kan extension** of F along G is a functor

$$G_! F \colon \mathscr{B} \to \mathscr{C}$$

defined on objects by the colimit

$$G_! F(b) = \operatorname*{colim}_{(G \downarrow b)} F \circ j_b,$$

where $(G \downarrow b)$ is the comma category of objects in $\mathscr{C}$ over b and $j_b \colon (G \downarrow b) \to \mathscr{C}$ forgets the morphism to b (to simplify the notation we will often omit writing j_b). Left Kan extensions possess a universal property: the functor $G_! F$ comes with a natural transformation

$$F \Rightarrow G_! F \circ P$$

that is initial among natural transformations from F to functors factoring through P.

If $\mathscr{C}$ is a Quillen model category (such as topological spaces or chain complexes) then there is a homotopy invariant (or, derived) version known as the **homotopy left Kan extension** $\mathbb{L}G_! F$; it is given by the formula

$$\mathbb{L}G_! F(b) = \operatorname*{hocolim}_{(G \downarrow b)} F \circ j_b.$$

This construction is homotopy invariant in the following sense: a natural transformation $F \to F'$ that is a pointwise homotopy equivalence induces a natural transformation $\mathbb{L}G_! F \to \mathbb{L}G_! F'$ that is also a pointwise homotopy equivalence. In fact, this is the left derived functor of left Kan extension with respect to the projective model structure on the functor categories.

There is a homotopy coherent version of the universal property for homotopy left Kan extensions. See [5, Proposition 6.1] for the details.

Note that there is a "Fubini theorem" for both ordinary and homotopy colimits,

$$\operatorname*{colim}_{\mathscr{A}} F \cong \operatorname*{colim}_{\mathscr{B}} G_! F \quad \text{and} \quad \operatorname*{hocolim}_{\mathscr{A}} F \xrightarrow{\sim} \operatorname*{hocolim}_{\mathscr{B}} \mathbb{L}G_! F.$$

3.7 Homotopy colimits, the Grothendieck construction and Thomason's Theorem

At several points we shall be taking homotopy colimits of diagrams in $\mathscr{T}op$ obtained from diagrams in $\mathscr{C}at$ by applying the classifying space

functor B (i.e. geometric realisation of the nerve) pointwise. Here we briefly recall a couple of useful tools for this situation.

Given a functor $F \colon \mathscr{C} \to \mathscr{C}at$, the **Grothendieck construction** on F, denoted $\int_{\mathscr{C}} F$ is the category in which objects are pairs ($x \in \mathscr{C}, y \in F(x)$), and a morphism $(x, y) \to (x', y')$ consists of an arrow $f \in \hom_{\mathscr{C}}(x, x')$ and an arrow $g \in \hom_{F(x')}(f_* y, y')$. This construction satisfies an associativity condition: if $F \colon \mathscr{C} \to \mathscr{C}at$ and $G \colon \int_{\mathscr{C}} F \to \mathscr{C}at$ are functors then sending $c \in \mathrm{Obj}\mathscr{C}$ to $\int_{F(c)} G$ defines a functor $\int_F G \colon \mathscr{C} \to \mathscr{C}at$ and there is a natural equivalence of categories

$$\int_{\int_{\mathscr{C}} F} G \simeq \int_{\mathscr{C}} \left(\int_F G \right).$$

Thomason's Theorem [11, Theorem 1.2] asserts that there is a natural homotopy equivalence,

$$\operatorname*{hocolim}_{\mathscr{C}} BF \xrightarrow{\simeq} B\left(\int_{\mathscr{C}} F \right).$$

As a special case, if $\mathscr{C}$ is actually a group G (a category with a single object $*$ and all arrows invertible), then $BF(*)$ is a space with a G action, and $B(\int_G \mathscr{F})$ is homotopy equivalent to the homotopy quotient $(B\mathscr{F}(*))_{hG}$.

If $\mathscr{C} = \Delta_{semi}^{op}$ then F is a semi-simplicial category, BF is a semi-simplicial space, and $B(\int_{\Delta_{semi}^{op}} F) \simeq \operatorname{hocolim} BF$ is equivalent to the geometric realisation of this semi-simplicial space.

There is a 2-categorical version of the above. First of all, given a strict 2-category $\mathscr{C}$ and a strict 2-functor $F \colon \mathscr{C} \to \mathscr{C}at^2$ there is a Grothendieck construction that produces a strict 2-category $\int_{\mathscr{C}} F$ over $\mathscr{C}$. An object of $\int_{\mathscr{C}} F$ is a pair ($x \in \mathrm{Obj}\mathscr{C}, y \in \mathrm{Obj}F(x)$). A 1-morphism $(x, y) \to (x', y')$ is a pair (f_1, f_2). where $f_1 \colon x \to x'$ is a 1-morphism in $\mathscr{C}$ and $f_2 \colon F(f_1)(y) \to y'$ is a morphism in $F(x')$. A 2-morphism $(f_1, f_2) \to (g_1, g_2)$ consists of a 2-morphism $\alpha \colon f_1 \Rightarrow g_1$ in $\mathscr{C}$ (which gives a natural transformation α_* from $F(f_1)$ to $F(g_1)$) such that the diagram (in $F(x')$)

$$
\begin{array}{ccc}
F(f_1)(y) & \xrightarrow{f_2} & y' \\
\downarrow{\scriptstyle \alpha_*} & \nearrow{\scriptstyle g_2} & \\
F(g_1)(y) & &
\end{array}
$$

commutes. This 2-categorical Grothendieck construction satisfies the obvious analogue of the associativity condition satisfied by the 1-categorical construction. Also, a 2-functor $\mathscr{C} \to \mathscr{T}op^2$ has a homotopy colimit, and a 2-categorical version of Thomason's theorem holds [2]: if $F : \mathscr{C} \to \mathscr{C}at^2$ is a 2-functor then

$$\operatorname*{hocolim}_{\mathscr{C}} BF \simeq B \int_{\mathscr{C}} F.$$

We again refer the reader to [2] and the references there for further details.

3.8   Graphs and Costello's graph category

For us, a *graph* Γ will consist of a set V of vertices, a set H of half-edges, an incidence map in : $H \to V$, and an involution $\iota : H \to H$, called the edge flip, that specifies how the half-edges are glued together. The free orbits of ι are the *edges* of the graph, denoted $E(\Gamma)$, so each edge consists of a pair of half-edges. The fixed points are called the *legs* and are denoted $L(\Gamma)$. The incidence map sends each half-edge to the vertex that it meets. A graph Γ has a topological realisation $|\Gamma|$ as a 1-dimensional CW complex with a 0-cell for each vertex and a 1-cell for each edge and leg. A graph is a *tree* if its topological realisation is contractible, and a *forest* if it is a union of trees.

A *corolla* is a graph that consists of a single vertex and a number of legs incident at it. If a graph is a disjoint union of corollas then the edge flip map is the identity and so giving a union of corollas is equivalent to giving a triple $(V, H, \mathrm{in} : V \to H)$. Associated with a graph Γ are two disjoint unions of corollas. The first is given by forgetting the edge flip and is denoted $\nu(\Gamma)$ (cutting each edge into a pair of legs). The second is denoted $\pi_0\Gamma$; it has one vertex for each connected component of the graph and one leg for each leg of the original graph Γ.

Costello [8] introduced a category $\mathscr{G}raphs$ in which the objects are disjoint unions of corollas and morphisms are given by graphs. In intuitive terms, we think of a morphism as assembling a bunch of corollas into a graph Γ followed by contracting all edges so that what remains is again a union of corollas (the result is $\pi_0\Gamma$). Composition of morphisms is defined by iterating this process. More precisely, the objects are triples, $(V, H, \mathrm{in} : V \to H)$; a morphism from $(V_1, H_1, \mathrm{in}_1 : V_1 \to H_1)$ to $(V_2, H_2, \mathrm{in}_2 : V_2 \to H_2)$ is represented by a graph Γ together with an isomorphism from the source to $\nu(\Gamma)$ and an isomorphism from the target to $\pi_0\Gamma$. There is an obvious notion of equivalence on these data and the

set of morphisms is defined as the set of equivalence classes. Alternatively, one can describe the set of morphisms as follows. A morphism consists of involution a on H_1 so that $(V_1, H_1, \text{in}_1 : V_1 \to H_1, a)$ defines a graph Γ together with an isomorphism of $\pi_0\Gamma$ with the union of corollas corresponding to the target. To define the composition of morphisms we use this second description. A composable pair of morphisms is given by a union of corollas, an involution a_1 on the half edges, and then a second involution a_2 on the set of fixed points of the first. The composition is given by the involution that is equal to a_1 on the free orbits of a_1 and is equal to a_2 on the fixed points of a_1.

Graphs also form a category in a different way, where the objects are graphs and the morphisms are given by contracting a set of tree subgraphs to points.

Disjoint union makes $\mathscr{G}raphs$ into a symmetric monoidal category. We will be interested in the symmetric monoidal subcategory $\mathscr{F}orests \subset \mathscr{G}raphs$ which has only those objects containing no 0-valent components (i.e., no isolated vertices) and only those morphisms that are forests (i.e., disjoint unions of trees); the inclusion functor will be denoted ℓ. We will also be interested in the over category of this inclusion. Fix a union of corollas x and consider the over category $\ell \downarrow x$.

Proposition 3.5. *The category $\ell \downarrow x$ is canonically equivalent to the category whose objects are graphs with legs identified with the legs of x and whose morphism are given by contracting a collection of disjoint trees down to points.*

3.9 Coloured graphs

Fix a set Λ of colours. A Λ-coloured graph is a graph together with an element of Λ assigned to each edge and each leg. One can form a category of Λ-coloured graphs, generalised Costello's category $\mathscr{G}raphs$, in which the objects are disjoint unions of corollas and the morphisms are coloured graphs.

3.10 Cyclic and modular operads as functors

Costello introduced his categories of graphs in order to reformulate the definition of cyclic and modular operads in terms more amenable to doing homotopy theoretic constructions.

Proposition 3.6. *The category of cyclic operads in $\mathscr{C}$ is equivalent to the category of symmetric monoidal functors $\mathscr{F}orests \to \mathscr{C}$, and*

the category of modular operads in $\mathscr{C}$ is equivalent to the category of symmetric monoidal functors $\mathscr{G}raphs \to \mathscr{C}$.

The idea is that a cyclic (or modular) operad $\mathcal{P}$ determines a functor by sending the n-corolla $*_n$ to the space $\mathcal{P}(n)$, and it sends a disjoint union of several corollas $*_{n_1} \sqcup \cdots \sqcup *_{n_k}$ to the product $\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k)$. Gluing legs together is sent to the map induced by the corresponding composition map.

Coloured cyclic and modular operads can of course also be described as symmetric monoidal functors, using the categories of coloured graphs.

3.11   Some examples: the commutative and associative operads

We will mainly be concerned with the case when the ambient category in which our operads live is the category $\mathscr{T}op$ of topological spaces.

The *commutative operad* $\mathcal{C}omm$ is the cyclic operad that is the constant functor $\mathscr{G}r \to \mathscr{T}op$ sending each graph to a single point $*$. Clearly the commutative operad can also be considered as a modular operad.

A *ribbon structure* on a graph is a choice of cyclic ordering of the half-edges incident at each vertex. A graph with ribbon structure is called a *ribbon graph*. The *associative operad* $\mathcal{A}ssoc$ is the cyclic operad that sends each graph γ to the discrete space consisting of one point for each ribbon structure on γ. It is not hard to see that if $\gamma \to \gamma'$ is a contraction of a tree subgraph then there is a canonical bijection between ribbon structures on γ and on γ'. There is also a canonical bijection between ribbon structures on γ and on its atomisation, and this provides the natural isomorphism required in the definition of a cyclic operad.

3.12   Modular envelopes

Restriction from $\mathscr{G}raphs$ to $\mathscr{F}orests$ defines a forgetful functor from modular operads to cyclic operads. This functor admits a left adjoint, Mod, called the *modular envelope*. We think of the modular envelope of a cyclic operad as the modular operad it freely generates. The modular envelope can be constructed via left Kan extension along the inclusion $\mathscr{F}orests \hookrightarrow \mathscr{G}raphs$.

By replacing the Kan extension with the derived Kan extension, we have the derived modular envelope functor $\mathbb{L}\mathrm{Mod}$.

4 Moduli of geometric structures on surfaces

4.1 Surfaces with collars

Let Σ be a surface with boundary and corners (by *corners*, we mean that it is locally modelled on the positive quadrant $[0,\infty)^2 \subset \mathbb{R}^2$). The boundary of Σ is canonically partitioned into smooth strata, each of which is either a circle of an interval. A *boundary interval* is a boundary stratum that is an interval. Let $J \subset \Sigma$ be a boundary interval. A *collar* of J is a smooth embedding ϕ of $(-1,0] \times [0,1]$ into Σ that sends boundary to boundary and is a diffeomorphism of $\{0\} \times [0,1]$ onto J. A surface equipped with a finite set of disjoint boundary intervals equipped with disjoint collars is called a *collared surface*. If the collars are labelled by a set I then we call the surface I-*collared*.

Suppose Σ_1 and Σ_2 are I-collared surfaces. A diffeomorphism of I-collared surfaces $\Sigma_1 \to \Sigma_2$ is a diffeomorphism of the underlying surfaces that respects the labelling and parametrization of the collars.

Suppose Σ is a surface with disjoint boundary intervals J_1 and J_2 equipped with disjoint collars ϕ_2 and ϕ_2 respectively. One can glue these two boundary intervals together and obtain a new smooth surface. This is done as follows: let $\Sigma' = \Sigma \smallsetminus (J_1 \cup J_2)/\sim$, where we identify $\phi_1(x)$ with $\phi_2(x)$ for each $x \in (-1,0) \times [0,1]$.

4.2 Sheaves of geometric structures

Let $\mathscr{S}urf$ be the category enriched in $\mathscr{T}op$ of finite type surfaces (possibly with boundary and corners) and open embeddings.

Definition 4.1. A *smooth sheaf* ψ on $\widetilde{\mathscr{S}urf}$ is an enriched functor $\widetilde{\mathscr{S}urf} \to \mathscr{T}op$ that sends pushout squares to homotopy pullback squares.

Remark 4.2. Smooth sheaves of this type have been studied in [3], where they are called *homotopy sheaves* and their relation with Goodwillie-Weiss embedding calculus of functors is explored. This notion also could go under the name of ∞-stacks.

Here we will think of the space $\psi(\Sigma)$ as the space of geometric structures of a given type on Σ. Below is a list of interesting examples of some of the kinds of structures that one can consider within this definition.

Example 4.3.   1. Orientations: $\psi(\Sigma)$ is the set of orientations on Σ.

2. Almost complex structures: Since the space of almost complex structures (for a fixed orientation) is contractible, this is equivalent to simply taking orientations.

3. Principal G-bundles: The space associated with a surface Σ is the space of maps $\Sigma \to BG$.

4. Maps to a background space X: The space $\psi(\Sigma)$ is the space of maps $\Sigma \to X$.

5. Spin, Spin^c and r-spin can all be described in terms of the space of lifts of the classifying map $\Sigma \to BSO(2)$ of the tangent bundle.

6. Foliations: $\psi(\Sigma)$ is the geometric realization of the simplicial space whose space of p-simplices is the space of codimension 2 foliations of $\Sigma \times \Delta^p$ that are transverse to the boundary of the simplex.[1]

One way to produce examples of smooth sheaves is to take sections of a bundle that is functorially associated with the tangent bundle. Let X be a space with an action of $GL_2(\mathbb{R})$. Given a surface Σ, let $P \to \Sigma$ be the $GL_2(\mathbb{R})$-principal bundle associated with the tangent bundle and consider the bundle $P \times_{GL_2(\mathbb{R})} X \to \Sigma$.

Proposition 4.4. *Sending Σ to the space of sections of $P \times_{GL_2(\mathbb{R})} X$ (with the compact-open topology) defines a smooth sheaf ψ_X on $\mathscr{S}urf$. Similarly, if X is a smooth manifold on which $GL_2(\mathbb{R})$ acts smoothly, then sending Σ to the space of smooth sections (with the smooth topology) defines a smooth sheaf.*

Those smooth sheaves arising in this way will be called *tangential*.

Remark 4.5. *A priori*, the definition of a smooth sheaf appears more general than the definition of tangential smooth sheaf. Not every smooth sheaf is tangential, such as the example of foliations in the list above. However, every smooth sheaf admits a tangnetial approximation and sometimes the approximation is actually equivalent to the original sheaf. In more detail, as described in [4, p. 16–17], given an smooth sheaf ψ, there is associated a tangential sheaf $\tau\psi$ and a canonical comparison morphism $\psi \to \tau\psi$. Moreover, (a version of) Gromov's h-principle gives conditions under which this comparison morphism is a weak equivalence of sheaves.

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[1]The author thanks the referee for suggesting the inclusion of this example.

A smooth sheaf ψ will be called *connected* if $\psi((-1,0) \times I)$ is connected. Assuming ψ is connected, we can choose a basepoint $* \in \psi((-1,0) \times I)$. Suppose $J \subset \partial\Sigma$ is a boundary interval equipped with a collar ϕ. We say that a section $s \in \psi(\Sigma)$ is *trivial at J* if $\phi^*(s)$ restricts to the chosen basepoint $*$ on $(-1,0) \times I$.

Remark 4.6. In proving the main theorem of this paper, the assumption that ψ is connected can be discarded if one is willing to work with coloured cyclic and modular operads instead of ordinary (single colour) cyclic and modular operads.

If Σ is a surface equipped with a collection of disjoint collared boundary intervals $\{J_1, \ldots, J_n\}$, we write

$$\widetilde{\psi}(\Sigma) \subset \psi(\Sigma)$$

for the subspace consisting of sections that are trivial at the boundary intervals J_i.

4.3 The monoid of geometric structures on a strip

Consider the unit square $I \times I$ equipped with a collar at each of the intervals $\{0\} \times I$ and $\{1\} \times I$ oriented in the same direction. We write A_ψ for the space $\psi'(I \times I)$ of sections that are trivial at each side of the square because this space will play a particularly important role in the results ahead.

Proposition 4.7. *Gluing squares side to side endows the space A_ψ with an A_∞ composition making it into a group-like A_∞ monoid; the homotopy inverse map is induced by rotating the square 180 degrees. Fixing a collared boundary interval J on a surface Σ, there is a right A_∞ action of A_ψ on $\widetilde{\psi}(\Sigma)$ by gluing the right side of a square to J, and a left A_∞ action given gluing the left edge of a square to J.*

We will not spell out the proof of this here; it is straightforward but technical because of the necessity of using some machinery to handle A_∞ monoids and their actions.

Proposition 4.8. *Let J_1 and J_2 be two disjointly collared boundary intervals on a surface Σ, and let Σ' be the result of gluing J_1 to J_2. There is a homotopy equivalence*

$$\psi(\Sigma') \sim \widetilde{\psi}(\Sigma)_{hA_\psi}$$

where the action of A_ψ is as follows. Given a square with ψ structure $K \in A_\psi$, we glue the left edge of one copy of K to J_1 and glue the left edge of a second copy of K to J_2.

This proposition is the key topological input in our generalised ribbon graph. The proof is rather technical and so it will be postponed for the future paper. The idea is straightforward and we explain it now. Gluing a square at J_1 has the effect of simply changing the trivialization of the ψ structure at J_1, and this action is transitive in a homotopical sense, so the homotopy quotient of this action is equivalent to the space of ψ-structures on Σ that are not necessarily trivial at J_1. Thus the homotopy quotient appearing in the proposition builds a model for the space of ψ-structures on Σ such that are not necessarily trivial at J_1 and J_2 but are required to agree at these collars. Giving such a structure is equivalent to giving a structure on the glued surface Σ'.

4.4   A 2-categorical model for the category of surfaces

In defining the modular operad of moduli spaces of ψ-structures, rather than the category $\widetilde{\mathscr{S}urf}$ of surfaces and open embeddings, we will need a slightly different category Let $\mathscr{S}urf$ denote the topological category whose objects are collared surfaces. In rough language, a morphism $\Sigma_1 \to \Sigma_2$ is a gluing of some collared boundary intervals together followed by a diffeomorphism. More precisely, the space of morphism is the disjoint union over all surfaces Σ' obtained from Σ by gluing a number of pairs of boundary intervals together of the space of diffeomorphisms $\Sigma' \to \Sigma_2$. We let $\mathscr{D}iscs \subset \mathscr{S}urf$ denote the full subcategory whose objects are disjoint unions of discs each having at least 1 collared boundary interval

These topological categories are difficult to work with, so it is convenient to replace them with more combinatorial models that will work just as well for our purposes. Let $\mathscr{S}urf^2$ and $\mathscr{D}iscs^2$ denote the strict 2-categories with the same objects as $\mathscr{S}urf$ and $\mathscr{D}iscs$ respectively, but with each space of diffeomorphisms replaced by the groupoid of diffeomorphisms and isotopy classes of isotopies.

Proposition 4.9. *Given a collared surface Σ, the nerve of the category $\mathrm{Hom}_{\mathscr{S}urf^2}(\Sigma, \Sigma)$ is weakly equivalent to the space $\mathrm{Hom}_{\widetilde{\mathscr{S}urf}}(\Sigma, \Sigma) \cong \mathrm{Diff}(\Sigma)$.*

Proof. When X is a disc or annulus with no collared boundary components then the diffeomorphism group is homotopy equivalent to a circle.

For any fixed diffeomorphism, there is a $\mathbb{Z}$ worth of isotopy classes of isotopies from it to itself, and the nerve of this $\mathbb{Z}$ gives the desired circle. In all other cases there is at most one isotopy class of isotopies between any two diffeomorphisms and the components of the diffeomorphism group are weakly contractible. □

Given a smooth sheaf ψ and a collared surface Σ, we have introduced the space $\widetilde{\psi}(\Sigma)$ of sections of ψ that are trivial at the collared boundary intervals. One sees that $\widetilde{\psi}$ determines a continuous functor $\mathscr{S}urf \to \mathscr{T}op$, which in turn determines a strict 2-functor $\mathscr{S}urf \to \mathscr{T}op^2$ that we shall denote by the same symbol.

In order to talk about cyclic and modular operads, we will need to versions of the above 2-categories in which the collared boundary intervals are labelled by a fixed finite set.

We define a strict 2-functor

$$S : \mathscr{G}raphs \to (\mathscr{C}at^2 \downarrow \mathscr{S}urf^2)$$

by sending a union of corollas τ to the strict 2-category of collared surfaces with components identified with the components of τ and collared boundary intervals compatibly identified with the legs of τ. The functor to $\mathscr{S}urf^2$ is given by forgetting the extra identification data. We also define

$$D : \mathscr{F}orests \to (\mathscr{C}at^2 \downarrow \mathscr{D}iscs^2)$$

analogously. One can check that these functors S and D are actually symmetric monoidal and hence they define cyclic and modular operads respectively (albeit in somewhat odd looking ambient categories). A value $S(\tau)$ of S consists of a 2-category $\mathscr{C}$ and a functor $P : \mathscr{C} \to \mathscr{S}urf$. Composing the $\widetilde{\psi}$ with P gives a functor which we will write (with a moderate abuse of notation) as $S(\tau) \to \mathscr{T}op^2$.

4.5   The modular operad of moduli spaces

Associated with a smooth sheaf ψ and a diffeomorphism type of surfaces $[\Sigma]$, there is a (homotopy theoretic) moduli space of surfaces diffeomorphic to Σ and equipped with a ψ-structure. This moduli space is simply the homotopy quotient

$$\psi(\Sigma)_{h\mathrm{Diff}(\Sigma)}.$$

If we consider collared surfaces equipped with ψ-structures that are trivial at the collared intervals then these moduli spaces collectively form

a modular operad. However, rigorously defining this modular operad so that it is strictly associative is somewhat subtle. As a first approximation, given a finite set I, the corresponding space of the modular operad is the disjoint union

$$(1) \qquad\qquad \coprod_\Sigma \widetilde{\psi}(\Sigma)_{h\mathrm{Diff}(\Sigma)}$$

where Σ runs over a set of representatives of diffeomorphism classes of surfaces equipped with a set of disjoint collared boundary intervals labelled by I. The composition maps $\circ_{i,j}$ and $_i\circ_j$ should be induced by gluing collared boundaries. However, with this construction, the composition maps would only be associative up to A_∞ homotopy.

One way to resolve this issue and strictify the composition maps is to use *all* surfaces (within some set-theoretic universe) rather than selecting one representative from each diffeomorphism class. To this end we make the following definition.

Definition 4.10. The moduli space modular operad $\mathcal{M}_\psi$ associated with a connected smooth sheaf ψ is defined by sending an object $\tau \in$ Obj $\mathscr{G}raphs$ (i.e., a union of corollas) to the homotopy colimit

$$\mathcal{M}_\psi(\tau) = \operatorname*{hocolim}_{S(\tau)} \widetilde{\psi}.$$

Similarly, we define a cyclic operad $\mathcal{D}_\psi$ of moduli spaces of discs by

$$\mathcal{D}_\psi(\tau) = \operatorname*{hocolim}_{D(\tau)} \widetilde{\psi}.$$

(To make sense of these symbols, please recall the abuse of notation mentioned above at the end of the previous subsection.)

In light of Proposition 4.9 above, $\mathcal{M}_\psi$ evaluated on a corolla with I legs yields a space homotopy equivalent to the homotopy quotient the the homotopy-theoretic moduli space of (1).

We can now state our main theorem.

Theorem 4.1. *The derived modular envelope of the cyclic operad $\mathcal{D}_\psi$ is weakly equivalent as a modular operad to $\mathcal{M}_\psi$.*

5   Arc systems in a surface

In this section we shall prove that the space of decompositions of a surface into discs is contractible.

5.1 Arcs and cutting

Let Σ be a collared surface with nonempty boundary. An *arc* in Σ is an embedding $[0,1] \to \Sigma$ that sends the interior of the interval to the interior of the surface, sends the endpoints to the boundary, and meets the boundary transversally. We consider two arcs to be equivalent if they differ by reversing the direction of the interval via $t \mapsto 1 - t$. If Σ is equipped with any collars then we require that the arc is disjoint from the collars. An *arc system* is a finite (possibly empty) collection of disjoint arcs in Σ that divide the surface in regions homeomorphic to discs (the regions will be diffeomorphic to polygons rather than discs since they will have corners), each of which touches at least one arc or collared boundary component.

Observe that a surface Σ equipped with an arc system A can by cut along the arcs of an arc system to yield a disjoint union of discs. Moreover, each of the boundary intervals created by the cutting can be collared uniquely up to diffeomorphism so that the resulting union of discs can be considered as a collared surface with one collared boundary interval for each collared boundary interval on the original surface plus one for each arc in the arc system. We shall denote this union of collared discs by Σ_A.

An arc system has a *dual graph* with one vertex for each region in the complement of the arcs, an edge for each arc, and a leg for each collar on the surface. We say that an arc system is *reduced* if its dual graph has the minimum possible number of bivalent vertices (1 in the case of a disc or annulus and zero in all other cases). An orientation on the surface induces a ribbon structure on the graph (i.e., a cyclic ordering of the half edges incident at each vertex).

5.2 The category of arc systems

A diffeomorphism of Σ sends arc systems to arc systems. An *isotopy* from an arc system A to an arc system B is a 1-parameter family of arc systems A_t such that $A_0 = A$ and $A_1 = B$. A bijection from the arcs of A to the arcs of B is said to be *admissible* if it can be induced by an isotopy.

Arc systems form a category $\mathscr{A}(\Sigma)$: the objects are arc systems and a morphism $A \to B$ consists an isotopy class of isotopies from A to a subsystem of B. We let $\mathscr{A}^r(\Sigma)$ denote the full subcategory of reduced arc systems. There is a reduction functor $R : \mathscr{A}(\Sigma) \to \mathscr{A}^r(\Sigma)$ that is

defined by replacing each collection of parallel arcs with a single arc and deleting any arc that is parallel to a collared boundary interval (this is non canonical the case of a disc or annulus, but a choice can be made in order to define the functor).

Theorem 5.1. *The nerve of $\mathscr{A}(\Sigma)$ is contractible.*

Proof. The proof is divided into three cases. (1) Σ is a disc without collars. (2) Σ is an annulus without collars. (3) Σ is any other collared surface.

(1) The dual graph of an arc system in the disc is a planar tree, and the set of univalent vertices (corresponding to arcs that bound discs containing no other arcs) inherits a cyclic order from the disc. Let Λ be the category of finite nonempty cyclically ordered sets and degree 1 maps (Connes's cyclic category). There is a functor $q : \mathscr{A} \to \Lambda$ given by sending an arc system to the set of univalent vertices of its dual graph. The nerve of Λ is known to be equivalent to BS^1, and we will identify the map induced by the above functor with the map $ES^1 \to BS^1$.

First, we show that the homotopy fibre of the map is S^1. For any object $[n] \in \Lambda$, consider the over category $q \downarrow [n]$. Let $\mathbb{Z}$ denote the category with a single object and a $\mathbb{Z}$ worth of endomorphisms. There is a functor $r : \mathbb{Z} \to (q \downarrow [n])$ given by sending the single object to the arc system consisting of a single arc and sending the generating automorphism to the automorphism given by rotating the disc through 360 degrees. Over any object of $q \downarrow [n]$, the over category of the functor r has a nonempty set of objects and by unwinding the definitions carefully one can see that there is a unique isomorphism between any two objects. Thus the nerve of any over category of r is contractible and Quillen's Theorem A implies that r induces a homotopy equivalence of nerves. By considering a generator of the fundamental group one can check that any morphism $[n] \to [m]$ in Λ induces a translation functor $(q \downarrow [n]) \to (q \downarrow [m])$ that is a homotopy equivalence on nerves. Quillen's Theorem B thus says that $B\mathbb{Z} \to \mathscr{A} \to \Lambda$ gives a homotopy fibre sequence upon passing to nerves. The nerve of the fibre is S^1 and the nerve of the base is BS^1.

To conclude that the total space is ES^1, we need only check that the inclusion of the fibre S^1 into the total space is trivial on π_1. The generator of π_1 of the fibre is represented by a rotation of the disc through 360 degrees. Given a symmetric configuration of 3 arcs in the disc, there is an automorphism given by rotation by 120 degrees, and the cube of this automorphism is the 360 degree rotation. A calculation in
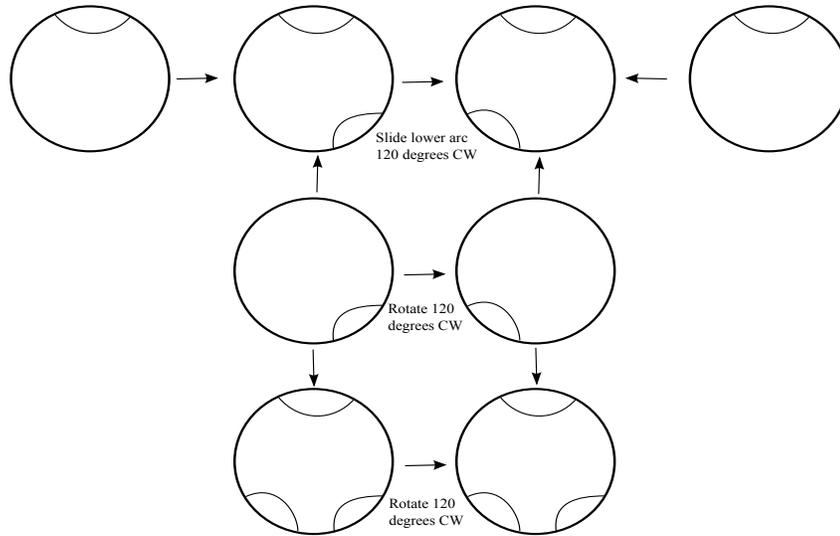
Figure 1: Calculation in the localised category representing the total space, showing the the generator of the fundamental group of the fibre is trivial in the total space. The unlabelled arrows in this diagram are all the evident inclusions of arc systems.

the localised category using §3.2 will show that this 120 degree rotation is trivial in the localisation. The calculation is represented by the diagram in figure 5.2.

(2) The dual graph of an arc system in the annulus is a chain of bivalent vertices (corresponding to those arcs which go from the inner boundary to the outer one) with some trees attached (corresponding to nested sets of arcs that have both ends on the same boundary circle). This chain on bivalent vertices inherits a cyclic order from the annulus, and sending an arc system to this set defines a functor from $\mathscr{A}$ to Λ. Using arguments similar to case (1) above, one can conclude that the homotopy fibre of the map of nerves is S^1 and the fibre sequence is in fact $S^1 \to ES^1 \to BS^1$.

(3) In this case we use a category-theoretic reformulation of an argument from Hatcher [12]. Let $\overline{\mathscr{A}}$ denote the category of isotopy classes of arc systems and admissible bijections. Since, in this case, every admissible bijection is induced by a unique isotopy class of isotopies, the canonical functor $\mathscr{A} \to \overline{\mathscr{A}}$ is an equivalence of categories.

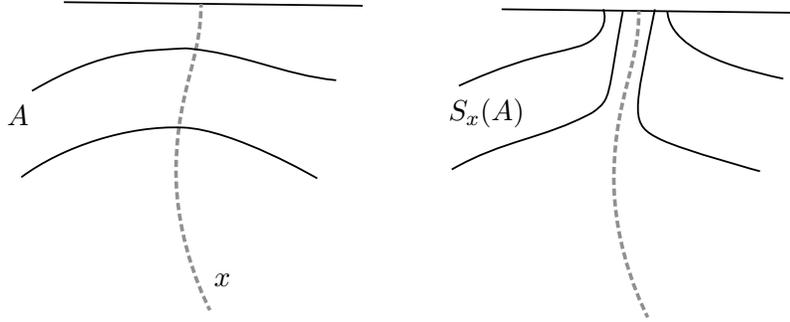Fix an arc x. We will show that the identity functor on $\mathscr{A}$ and

Figure 2: The effect of the arc surgery operator S_x.

the constant functor sending any arc system to x are homotopic after passing to nerves by constructing a zigzag of natural transformations between functors $\mathscr{A} \to \mathscr{A}$, starting with the identity and finishing with the constant functor.

We define an operator S_x from arcs to arcs as follows. If x and y are disjoint (up to isotopy) then $S_x(y) = y$. If x and y intersect then, moving them by isotopies, we may put them in a position so that they cross transversally and the number of intersection points is minimal (e.g., choose a metric and use the geodesic representatives of their isotopy classes). We may now cut y at each point where it meets x and slides the resulting endpoints along x until they reach the boundary of the surface, as shown in figure 5.2. We can extend S_x to a map from arc systems to arc systems by applying it to each of the arcs in a system.

Let $S_1 : \mathscr{A} \to \mathscr{A}$ be the functor that sends an arc system A to $A \cup S_x(A)$, let S_2 be the functor that sends A to $S_x(A)$, let S_3 be the functor that sends A to $x \cup S_x(A)$, and let S_4 be the constant functor sending any arc system to x. It is straightforward to see that there are natural transformations

$$\mathrm{id} \to S_1 \leftarrow S_2 \to S_3 \leftarrow S_4$$

induced by the evident inclusions of arc systems. Upon passing to nerves, this zigzag of natural transformation yields the desired homotopy from the identity map to the constant map. $\qquad\square$

Corollary 5.1. *The nerve of $\mathscr{A}^r(\Sigma)$ is contractible.*

Proof. Let i denote the inclusion $\mathscr{A}^r \hookrightarrow \mathscr{A}$ and observe that for any arc system A the over category $i \downarrow A$ has an initial object given by

any choice reduction $R(A)$ and morphism $R(A) \to A$. The result then follows from Quillen's Theorem A and Theorem 5.1. $\qquad\square$

6 Sketch of the proof Theorem 4.1

Let ψ be a smooth sheaf and recall that we have defined the cyclic operad $\mathcal{D}_\psi$ of moduli spaces of discs, and the modular operad $\mathcal{M}_\psi$ of moduli spaces of surfaces. We will construct a chain of weak equivalences between the derived modular envelope $\mathbb{L}\mathrm{Mod}(\mathcal{D}_\psi)$ and $\mathcal{M}_\psi$.

By the construction of the derived Kan extension along $\ell : \mathscr{F}orests \hookrightarrow \mathscr{G}raphs$ (evaluated at τ as the homotopy colimit over the over category $\ell \downarrow \tau$, we see that the derived modular envelope of $\mathcal{D}_\psi$, evaluated on a union of corollas γ, is given by

$$\operatorname*{hocolim}_{\gamma \in \ell \downarrow \tau} \mathcal{D}_\psi = \operatorname*{hocolim}_{\gamma \in \ell \downarrow \tau} \ \operatorname*{hocolim}_{D(\gamma)} \widetilde{\psi}.$$

By the Fubini theorem for homotopy colimits, this is weakly equivalent to $\operatorname{hocolim}_{\int_{\ell \downarrow \tau} D} \widetilde{\psi}$.

An object of $\int_{\ell \downarrow \tau} D$ is a graph γ (equipped with an identification $\pi_0 \gamma \cong \tau$) and a decoration of each vertex by a disc. Gluing the discs together as prescribed by the graph results in a surface $\Sigma \in S(\tau)$. Moreover, the collection of arcs in Σ along which the gluing was performed yield an arc system. This defines a strict 2-functor.

Lemma 6.1. *The above construction defines an equivalence of 2-categories $\int_{\ell \downarrow \tau} D \simeq \int_{S(\tau)} \mathscr{A}$.*

The inverse of the equivalence is constructed by cutting along the arcs, and it is denoted κ. Note that it is *not* a strict 2-functor but only a lax 2-functor. Hence

$$\operatorname*{hocolim}_{\int_{\ell \downarrow \tau} D} \widetilde{\psi} \simeq \operatorname*{hocolim}_{\int_{S(\tau)} \mathscr{A}} \widetilde{\psi} \circ \kappa,$$

which is weakly equivalent to

$$\operatorname*{hocolim}_{\int_{S(\tau)} \mathscr{A}^r} \mathbb{L}R_!(\widetilde{\psi} \circ \kappa),$$

where $R : \int_{S(\tau)} \mathscr{A} \to \int_{S(\tau)} \mathscr{A}^r$ is the arc system reduction functor.

We now come to the key step in the argument. As explained in [13], Kan extension along R can be thought of as integrating out the

bivalent vertices in the dual graphs to the arc systems, and this has the following effect. For each edge in the dual graph, one builds a 2-sided bar construction for the monoid A_ψ of geometric structures on a strip acting on the the spaces associated to the vertices at either end of the edge. By Proposition 4.8 we then have the following result.

Lemma 6.2. $\mathbb{L}R_!(\widetilde{\psi} \circ \kappa) \simeq \widetilde{\psi} \circ \pi$, *where* $\pi : \int_{S(\tau)} \mathscr{A}^r \to S(\tau)$ *forgets the arc system.*

In other words, starting with the space of all pairs of an unreduced arc system and a ψ-structure trivial on the arcs, and then integrating out the bivalent vertices in the dual graph yields a space equivalent to the space of pairs of a reduced arc system and a ψ-structure not-necessarily trivial on the arcs.

Finally, by Corollary 5.1, the homotopy colimit of $\widetilde{\psi} \circ \pi$ over $\int_{S(\tau)} \mathscr{A}^r$ is equivalent to the homotopy colimit of $\widetilde{\psi}$ over $S(\tau)$, which is precisely the definition of $\mathcal{M}_\psi(\tau)$.

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Jeffrey Giansiracusa
Department of Mathematics,
Swansea University,
Singleton Park, Swansea, Wales,
SA2 8PP, United Kingdom,
j.h.giansiracusa@swansea.ac.uk

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