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Homotopy theory of non-orientable mapping class groups

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Abstract

We give a homotopical approach to the theory of mapping class groups of surfaces with marked points. Using configuration spaces we construct Eilenberg-MacLane spaces $K(\pi, 1)$ for the mapping class groups of the projective plane P^2 and the Klein bottle $\mathbb{K}$. These spaces are closely related to the $K(\pi, 1)$ spaces for the corresponding braid groups. Some cohomological consequences of this approach are presented.

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1 Introduction

Let S_g be the compact orientable surface of genus g and let $\mathrm{Diff}^+(S_g)$ denote the topological group of orientation-preserving self-diffeomorphisms of S_g. The *mapping class group* of S_g is the group $\Gamma(S_g)$ of isotopy classes in $\mathrm{Diff}^+(S_g)$, that is, $\Gamma(S_g) = \pi_0\mathrm{Diff}^+(S_g)$. This group

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has been widely studied in the past few decades specially due to its action on the Teichmüller space $\mathcal{T}_g$ of isotopy classes of complex structures on S_g. The quotient by this action is the moduli space $\mathcal{M}_g$ of Riemann surfaces, since the space $\mathcal{T}_g$ is contractible and the action of $\Gamma(S_g)$ is properly discontinuous ([18]), there is an isomorphism on rational cohomology $H^*(\mathcal{M}_g;\mathbb{Q}) \cong H^*(\Gamma(S_g);\mathbb{Q})$. This isomorphism is one of the main motivations for studying the cohomology of mapping class groups.

On the other hand, knowing the cohomology groups of the mapping class group is also useful in the classification of surface bundles. A *surface bundle* or *S_g-bundle* is a fiber bundle where the fiber is S_g and the structure group of S_g-bundles is $\mathrm{Diff}^+(S_g)$. One way of determining if two S_g-bundles are isomorphic or not is considering characteristic classes, which are classes in $H^*(B\mathrm{Diff}^+(S_g);\mathbb{Z})$ which are natural with respect to bundle maps. It turns that for genus 0 and 1 this problem is solved (see [21]) and for genus $g \geq 2$ it turns out that $B\mathrm{Diff}^+(S_g)$ is the classifying space of $\Gamma(S_g)$ and thus characteristic classes of S_g-bundles are classes in $H^*(\Gamma(S_g);\mathbb{Z})$.

This work is concentrated on the homotopy theory associated with the theory of mapping class groups when the surface is non-orientable and has a set of k distinguished points. These groups are denoted by $\Gamma^k(S)$ and are called the *punctured* mapping class group of the surface S. Some $K(\pi,1)$ for these groups arise naturally from certain actions on the unordered configuration spaces $F_k(S)/\Sigma_k$. The motivation for this approach is the relation of configuration spaces with surface braid groups and thus some constructions applies to these groups.

Section 2 begins with the definition of the groups $\Gamma^k(S)$ and then a discussion on the construction of $K(\pi,1)$ spaces for these groups involving Borel constructions on configuration spaces. Section 3 focuses on the development of basic tools for the theory of configurations spaces which includes Fadell-Neuwirth fibrations and the identification of the fundamental group of $F_k(S)$ and $F_k(S)/\Sigma_k$ as braid groups of S. The case of P^2 is tackled in Section 4 using the theory of *orbit configuration spaces* $F_k(M;G)$ with a particular interest in the spaces $F_k(S^2;\mathbb{Z}_2)$ as they are related to $F_k(\mathrm{P}^2)$ via the covering map $S^2 \to \mathrm{P}^2$. This will be used to show that the $SO(3)$-Borel construction of $F_k(\mathrm{P}^2)/\Sigma_k$ is a $K(\Gamma^k(\mathrm{P}^2),1)$ space.

Configuration spaces of the Klein bottle $\mathbb{K}$ are considered in Section

5 to demonstrate that the space $ESO(2) \times_{SO(2)} F_k(\mathbb{K})/\Sigma_k$ is a $K(\pi, 1)$ space for $\widetilde{\Gamma}^k(\mathbb{K})$, where $\widetilde{\Gamma}^k$ denotes the *reduced* mapping class group defined in Section 2. Here the $SO(2)$-action is given by rotations on the first coordinate and is thought to be the restriction of the action of the group $\mathrm{Diff}_0(\mathbb{K})$ of diffeomophisms of $\mathbb{K}$ isotopic to the identity.

The work is concluded by recalling a method for calculating the additive structure of the cohomology of unordered configuration spaces $F_k(M)/\Sigma_k$ when M is a surface. This method consist on recognizing the homology of configuration spaces as a part of the homology of a larger space $C(M; X)$ called the *labelled configuration spaces* or *configuration space with parameters*, which is based on the classical work of C. F. Bödigheimer, F. Cohen and L. Taylor ([7]). This method is useful for obtaining information about the cohomological structure of mapping class groups, although the details of the calculations will appear elsewhere.

2 The punctured mapping class group

The mapping class group $\Gamma(S)$ of a compact surface S is the group of isotopy classes in $\mathrm{Diff}(S)$. Equivalently, one has $\Gamma(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$, where $\mathrm{Diff}_0(S)$ consists of self-diffeomorphisms of S which are isotopic to the identity. As mentioned, we are interested on diffeomorphisms preserving a set of distinguished points. Let $Q_k \subset S$ denote a subset of cardinality k and consider the group

$$\mathrm{Diff}^+(S; k) = \{f \in \mathrm{Diff}^+(S) \mid f(Q_k) = Q_k\}$$

The *punctured* mapping class group of S is the group $\Gamma^k(S)$ of isotopy classes of $\mathrm{Diff}^+(S; k)$, where the isotopies preserve the set Q_k. As above, one can also define $\Gamma^k(S) = \mathrm{Diff}(S; k)/\mathrm{Diff}_0(S; k)$. The punctured mapping class group arises from the study of the mapping class group of a surface and that of its branched covering ([4]).

From the equivalent definitions above it is easy to see that the problem of determining the homotopy type of both $\mathrm{Diff}(S)$ and $\mathrm{Diff}_0(S)$ plays an important role in the theory of mapping class groups. One finds that, apart from few cases, $\mathrm{Diff}_0(S)$ and $\mathrm{Diff}(S)$ do not have the same homotopy type, which will have a great impact on the constructions made in this work. We thus consider a slightly different definition of $\Gamma^k(S)$ and define the *reduced* mapping class group $\widetilde{\Gamma}^k(S)$ as the group of path components of $\mathrm{Diff}_0(S) \cap \mathrm{Diff}(S;k)$, which is $\widetilde{\Gamma}^k(S) = \pi_0(\mathrm{Diff}_0(S) \cap \mathrm{Diff}(S;k))$. This group fits into an exact sequence of the form ([20])

$$1 \longrightarrow \widetilde{\Gamma}^k(S) \longrightarrow \Gamma^k(S) \longrightarrow \Gamma(S) \longrightarrow 1$$

We mention some low-genus cases for this sequence. First, recall that the mapping class group $\Gamma(S^2)$ is trivial ([3]) since every orientation-preserving self-diffeomorphism of S^2 is isotopic to the identity. The sequence above gives the isomorphism $\widetilde{\Gamma}^k(S^2) \cong \Gamma^k(S^2)$. Now, since every self-diffeomorphism of P^2 can be lifted to a self-diffeomorphism of S^2 it follows that $\Gamma(\mathrm{P}^2)$ consists of a single class and there is an isomorphism $\widetilde{\Gamma}^k(\mathrm{P}^2) \cong \Gamma^k(\mathrm{P}^2)$.

Recall that the Klein bottle $\mathbb{K}$ can be obtained as the quotient of $T^2 = S^1 \times S^1$ by identifying a pair (u_1, u_2) with $(-u_1, \overline{u}_2)$, where $\overline{u}$ is complex conjugation. The mapping class group $\Gamma(\mathbb{K})$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ ([19]) and thus there is a sequence

$$1 \longrightarrow \widetilde{\Gamma}^k(\mathbb{K}) \longrightarrow \Gamma^k(\mathbb{K}) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

which exhibits $\widetilde{\Gamma}^k(\mathbb{K})$ as an order 4 subgroup of the punctured mapping class group.

As will be shown our $K(\pi, 1)$ constructions depend heavily on the homotopy type of $\mathrm{Diff}(S)$ and $\mathrm{Diff}_0(S)$, which is recorded next in a classical theorem from A. Gramain.

Theorem 2.1. *Let N be a compact connected surface, non-orientable, with or without boundary, and let $\mathit{Diff}_0(N)$ be the subgroup of $\mathit{Diff}(N)$ of diffeomorphisms isotopic to the identity. Then,*

1. *If N is P^2, then $\mathit{Diff}_0(\mathrm{P}^2) \simeq \mathit{Diff}(\mathrm{P}^2)$ is homotopy equivalent to $SO(3)$.*
2. *If N is the Klein bottle or the Möbius band, then $\mathit{Diff}_0(S)$ is homotopy equivalent to $SO(2)$.*
3. *For other non-orientable surfaces, $\mathit{Diff}_0(N)$ is contractible.*

A more general statement of this theorem involving orientable surfaces is considered in [12],[17], but for most surfaces $\mathrm{Diff}_0(S)$ is contractible. The rest of the paper will concentrate on the construction of $K(\pi,1)$ spaces for the groups $\Gamma^k(\mathrm{P}^2)$ and $\widetilde{\Gamma}^k(\mathbb{K})$, which will be done using the concept of *orbit homotopy space* or *Borel construction* recorded in the following.

Let G be a topological group acting on a space X. The *Borel construction* associated to this action is the quotient

$$EG \times_G X := (EG \times X)/G,$$

where EG is a contractible space with a free G-action and the G-action on the product is given by $g \cdot (e,x) = (eg^{-1}, gx)$. This construction appears in the field of equivariant cohomology where the ordinary cohomology of $EG \times_G M$ is defined as the *equivariant cohomology* of a manifold M with a G-action.

The present work focuses on the $\mathrm{Diff}(S)$-action on the configuration spaces $F_k(X)/\Sigma_k$ and the associated Borel constructions have a clear description using the Theorem 2.1 as we will show.

In the case of the group $\mathrm{Diff}(\mathrm{P}^2)$ there is a homotopy equivalence obtained from a diagram of fibrations

$$E\mathrm{Diff}(\mathrm{P}^2) \underset{\mathrm{Diff}(\mathrm{P}^2)}{\times} F_k(\mathrm{P}^2)/\Sigma_k \simeq ESO(3) \underset{SO(3)}{\times} F_k(\mathrm{P}^2)/\Sigma_k,$$

where the diagonal action of $SO(3)$ on $F_k(\mathrm{P}^2)/\Sigma_k$ is induced from the action by rotations of lines through the origin in $\mathbb{R}^3$. Also from Theorem 2.1 above there is a homotopy equivalence

$$E\mathrm{Diff}_0(\mathbb{K}) \underset{\mathrm{Diff}_0(\mathbb{K})}{\times} F_k(\mathbb{K})/\Sigma_k \simeq ESO(2) \underset{SO(2)}{\times} F_k(\mathbb{K})/\Sigma_k,$$

where the diagonal action of $SO(2)$ is given by rotations on the first coordinate of $\mathbb{K} = T^2/\approx$ defined above. Finally, note that for non-orientable surfaces of genus $g \geq 2$, there is a homotopy equivalence

$$E\mathrm{Diff}_0(N_g) \times_{\mathrm{Diff}_0(N_g)} F_k(N_g)/\Sigma_k \simeq F_k(N_g)/\Sigma_k.$$

In the next sections we consider the constructions associated to P^2 and $\mathbb{K}$ and we prove they are indeed $K(\pi,1)$ spaces for the groups $\Gamma^k(S)$.

This will be done using some tools from the theory of configuration spaces and basic properties of Borel constructions. These topics are introduced in the next section.

3 Surface configuration spaces

Let M be a closed manifold. For $k \geq 1$ define the *k-th configuration space of* M as the subspace of M^k given by

$$F_k(M) = \{(x_1, \dots, x_k) \mid x_i \neq x_j, \text{for } i \neq j\}$$

For $1 \leq m \leq k$ consider the projection

$$p_{k,m} : F_k(M) \longrightarrow F_m(M), \qquad (x_1, x_2, \dots, x_k) \mapsto (x_1, \dots, x_m),$$

and note that for a fixed $\hat{x} = (x_1, \dots, x_m)$ the space $p_{k,m}^{-1}(\hat{x})$ is identified with $F_{k-m}(M \backslash Q_m)$, where Q_m is a subset of cardinality m.

Theorem 3.1. *The projection $p_{k,m}$ is a fibration with fiber $F_{k-m}(M \backslash Q_m)$*

These maps are called the *Fadell-Neuwirth fibrations*, named after E. Fadell and L. Neuwirth who introduced the theory in the classical paper [13]. These fibrations represent one of the main tools in the theory of configuration spaces and are mainly used in inductive arguments.

It is classically known that the fundamental group of $F_k(\mathbb{R}^2)/\Sigma_k$ is isomorphic to the classical Artin braid group on k strands B_k and for ordered configurations $F_k(\mathbb{R}^2)$ one gets the *pure* braid group P_k ([14]). A basic analysis of these spaces shows these are Eilenberg-MacLane spaces $K(\pi, 1)$. In general the *pure braid group* of M is defined as $P_k(M) = \pi_1 F_k(M)$ and $B_k(M) = \pi_1 F_k(M)/\Sigma_k$ is its associated *braid group.*

The theory of surface configuration spaces is particularly interesting due to a result of J. Birman ([2]) which states that the homomorphism

$$i_* : P_k(M) \longrightarrow (\pi_1 M)^k$$

induced by the natural inclusion $i : F_k(M) \to M^k$ is an isomorphism if $\dim M \geq 3$ and is an epimorphism for $\dim M = 2$. This result shows that the structure of $P_k(M)$ is totally determined by the geometry of M (expressed by the fundamental group) and when M is a surface the group $P_k(M)$ exhibits a more complex structure since it has the pure braid group P_k as subgroup, arising from the canonical inclusion $\mathbb{R}^2 \subset M$. In fact, for a surface distinct from the 2-sphere and the projective plane, the kernel of the homomorphism i_* above is the normal closure of P_k ([16]). From a homotopical point of view more can be said

Theorem 3.2 ([13]). *For a compact 2-manifold M that is neither the 2-sphere S^2 nor the projective plane P^2, the spaces $F_k(M \backslash Q_m)$ and $F_k(M \backslash Q_m)/\Sigma_k$ are $K(\pi, 1)$ spaces, for $m \geq 0$.*

This theorem shows that from a homotopical point of view questions about configuration spaces of surfaces may be reduced to algebraic questions about their associated braid groups.

Configuration spaces for S^2 are not $K(\pi, 1)$ since $F_k(S^2)$ contains the higher homotopy of S^2 and $SO(3)$, see [10]. For configurations on P^2 the space $ES^3 \times_{S^3} F_k(\mathrm{P}^2)$ is a $K(\pi, 1)$ ([23]), where the S^3-action is induced from the double cover $S^3 \to SO(3)$. This construction is the total space of the fibration

$$F_k(\mathrm{P}^2) \longrightarrow ES^3 \underset{S^3}{\times} F_k(\mathrm{P}^2) \longrightarrow BS^3,$$

whose long exact sequence in homotopy shows that there is an isomorphism $\pi_n F_k(\mathrm{P}^2) \cong \pi_n(S^3)$, for $n \geq 2$ and thus $F_k(\mathrm{P}^2)$ is not a $K(\pi, 1)$ space. In order to obtain a $K(\pi, 1)$ space for the S^2 and P^2 cases one must consider a Borel construction on their associated configuration space.

Configuration spaces as homogeneous spaces. Note the group $\mathrm{Diff}(S)$ of self-diffeomorphisms of S acts transitively on the configuration space $F_k(S)/\Sigma_k$. That is, given two configurations $\hat{x}, \hat{y} \in F_k(S)/\Sigma_k$ there is an element $f \in \mathrm{Diff}(S)$ such that $f(\hat{x}) = \hat{y}$. Moreover, this diffeomorphism can be choosen to be isotopic to the identity ([20]). Also note that for a basepoint $\hat{x} \in F_k(S)/\Sigma_k$, the isotropy subgroup is precisely $\mathrm{Diff}(S; k)$.

On the other hand, recall that for a general compact manifold M the topological group $\mathrm{Diff}(M)$ is a metrizable manifold modeled in a

Fréchet space ([12], [1]); hence it has the homotopy type of a CW-complex. In particular, for a compact surface $\mathrm{Diff}(S)$ is locally compact and its diagonal action on $F_k(S)/\Sigma_k$ induces a homeomorphism $\mathrm{Diff}(S)/\mathrm{Diff}(S;k) \cong F_k(S)/\Sigma_k$ ([22]). Under these conditions the Borel construction $E\mathrm{Diff}(S) \times_{\mathrm{Diff}(S)} F_k(S)/\Sigma_k$ is homotopy equivalent to

$$E\mathrm{Diff}(S) \times_{\mathrm{Diff}(S)} \mathrm{Diff}(S)/\mathrm{Diff}(S;k) \simeq E\mathrm{Diff}(S)/\mathrm{Diff}(S;k),$$

where the space on the right is a model for the classifying space of $\mathrm{Diff}(S;k)$. Note that if we consider the fundamental group of this Borel construction we get

$$\pi_1(E\mathrm{Diff}(S) \times_{\mathrm{Diff}(S)} F_k(S)/\Sigma_k) \cong \pi_1 B\mathrm{Diff}(S;k) \cong \Gamma^k(S).$$

This is the motivation for considering Borel constructions for constructing Eilenberg-MacLane spaces for punctured mapping class groups. In the following sections we consider the case of the projective plane P^2 and the Klein bottle $\mathbb{K}$. In particular, we will prove that the spaces obtained at the end of Section 2 are indeed $K(\pi,1)$ spaces. We close this section showing a relation between braid groups and mapping class groups via covering maps.

Let $SO(3)$ acts on S^2 by rotations. For the diagonal action of $SO(3)$ on the space $F_k(S^2)/\Sigma_k$ one has the following

Proposition 3.3 ([9])**.** *For $k \geq 3$, $ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$ is an Eilenberg-MacLane space of the type $K(\Gamma^k(S^2),1)$.*

The universal cover $\widetilde{SO(3)}$ can be identified with the quaternions of unit length, which is the 3-sphere S^3. Thus the map $\phi : S^3 \to SO(3)$ induces a S^3-action on configurations of S^2.

Corollary 3.4. *For $k \geq 3$ the space*

$$ES^3 \underset{S^3}{\times} F_k(S^2)/\Sigma_k$$

is an Eilenberg-MacLane space of the type $K(B_k(S^2),1)$.

Here the $K(\pi, 1)$ part follows from the diagram of fibrations

$$\begin{tikzcd} X \arrow[r] \arrow[d, "id"'] & ES^3 \times_{S^3} X \arrow[r] \arrow[d] & BS^3 \arrow[d, "B\phi"] \\ X \arrow[r] & ESO(3) \times_{SO(3)} X \arrow[r] & BSO(3) \end{tikzcd}$$

and the following lemma

Lemma 3.5 ([23])**.** *If* $ESO(3) \underset{SO(3)}{\times} X$ *is a* $K(\pi, 1)$*, then* $ES^3 \underset{S^3}{\times} X$ *is a* $K(\pi', 1)$*. Moreover,* $\pi' \cong \pi_1(X)$*.*

4 Orbit configuration spaces

Let G be a group acting freely on a connected manifold M. The *k-th orbit configuration space* of M ([24]) is the space $F_k(M; G)$ of k-tuples of points on distinct G-orbits:

$$F_k(M; G) = \{(m_1, \ldots, m_k) \in M^k \mid Gm_i \cap Gm_j = \varnothing,\ i \neq j\},$$

where Gm denotes the G-orbit of m. Note that $F_1(M; G) = M$ since the action is free and for the trivial group e one has $F_k(M; e) = F_k(M)$.

In this context of group actions on manifolds there is also a version of the Fadell-Neuwirth fibrations.

Theorem 4.1. *For* $1 \leq m \leq k$*, the projection on the first* m *coordinates*

$$p_{k,m} : F_k(M; G) \longrightarrow F_m(M; G)$$

is a fibration with fiber $F_{k-m}(M \backslash O_m; G)$*, where* O_m *is the disjoint union of* m *distinct orbits.*

There is another tool relating the quotient M/G of the G-action and the orbit configuration spaces $F_k(M; G)$. For a principal G-bundle $\pi : M \to M/G$ there is an induced principal G^k-bundle

$$\tilde{\pi} : F_k(M; G) \longrightarrow F_k(M/G),$$

where $\tilde{\pi}(m_1, \ldots, m_k) = (\pi(m_1), \ldots, \pi(m_k))$ and the action of G^k on $F_k(M; G)$ is given by

$$(g_1, g_2, \ldots g_k) \cdot (m_1, m_2, \ldots, m_k) = (g_1 m_1, g_2 m_2, \ldots, g_k m_k).$$

For the proof of these statements see [24].

As an example consider $\mathbb{Z}_2$ acting on S^2 via the antipodal map. By the comments above, there is a covering map

$$(\mathbb{Z}_2)^k \longrightarrow F_k(S^2; \mathbb{Z}_2) \longrightarrow F_k(\mathrm{P}^2)$$

which shows that one way for studying configurations on the projective plane is considering the orbit configuration spaces for the 2-sphere. We will do this in the next paragraphs.

Let us consider the orbit configuration space for $S^2 \backslash O_n$ and note that

$$F_1(S^2 \backslash O_n; \mathbb{Z}_2) = S^2 \backslash O_n = S^2 \backslash Q_{2n},$$

where $O_n = Q_{2n}$ consists of n distinct pairs of antipodal points. We thus note that $F_1(S^2 \backslash O_n; \mathbb{Z}_2)$ is a $K(\pi, 1)$ space. Applying induction on k in the fibration

$$F_1(S^2 \backslash O_{n+k-1}; \mathbb{Z}_2) \longrightarrow F_k(S^2 \backslash O_n; \mathbb{Z}_2) \longrightarrow F_{k-1}(S^2 \backslash O_n; \mathbb{Z}_2)$$

we obtain that $F_k(S^2 \backslash O_n; \mathbb{Z}_2)$ is a $K(\pi, 1)$.

Theorem 4.2. *Let $SO(3)$ acts diagonally on $F_k(S^2; \mathbb{Z}_2)$. Then, for $k \geq 2$, the space $ESO(3) \underset{SO(3)}{\times} F_k(S^2; \mathbb{Z}_2)$ is a $K(\pi, 1)$.*

Proof. Consider the subspace W of $F_2(S^2; \mathbb{Z}_2)$ given by pairs of orthogonal vectors of unit lenght. The group $SO(3)$ acts on W transitively and freely and commutes with the antipodal action of $\mathbb{Z}_2$. Thus one has that $W = SO(3)$ and the inclusion

$$W \longrightarrow F_2(S^2; \mathbb{Z}_2)$$

is a $SO(3)$-equivariant homotopy equivalence ([10]). The inclusion thus induces a homotopy equivalence between Borel constructions

$$ESO(3) \underset{SO(3)}{\times} W \xrightarrow{\simeq} ESO(3) \underset{SO(3)}{\times} F_2(S^2; \mathbb{Z}_2).$$

The space on the left is homotopy equivalent to $ESO(3)$ which is contractible. Thus $ESO(3) \times_{SO(3)} F_2(S^2;\mathbb{Z}_2)$ is contractible.

On the other hand, consider the fibration

$$F_{k-2}(S^2\backslash O_2;\mathbb{Z}_2) \longrightarrow F_k(S^2;\mathbb{Z}_2) \xrightarrow{p} F_2(S^2;\mathbb{Z}_2).$$

By the comments preceding the theorem, the fiber is $K(\pi,1)$ and thus the map p induces isomorphisms

$$p_* : \pi_n F_k(S^2;\mathbb{Z}_2) \xrightarrow{\cong} \pi_n F_2(S^2;\mathbb{Z}_2), \quad n \neq 1$$

Finally, consider the commutative diagram

$$\begin{tikzcd}
F_k(S^2;\mathbb{Z}_2) \arrow[r] \arrow[d, "p"] & ESO(3) \underset{SO(3)}{\times} F_k(S^2;\mathbb{Z}_2) \arrow[r] \arrow[d] & BSO(3) \arrow[d, "id"] \\
F_2(S^2;\mathbb{Z}_2) \arrow[r] & ESO(3) \underset{SO(3)}{\times} F_2(S^2;\mathbb{Z}_2) \arrow[r] & BSO(3)
\end{tikzcd}$$

where each row is a fibration. Since p induces isomorphisms on π_n for $n \neq 1$ if follows that

$$\pi_n \left(ESO(3) \underset{SO(3)}{\times} F_k(S^2;\mathbb{Z}_2) \right) = 0$$

for $n \neq 1$ and the theorem follows. □

The main consequence of this theorem is a determination of the homotopy type of the Borel construction for the $SO(3)-$action on configurations of the projective plane. This result can also be proved directly from the identification of P^2 with the Grassmann manifold $O(3)/O(1) \times O(2)$, this proof is given in [23].

Theorem 4.3. *For $k \geq 2$, the construction $ESO(3) \times_{SO(3)} F_k(\mathrm{P}^2)$ is a $K(\pi,1)$ space.*

Proof. Given the covering map $\pi : F_k(S^2;\mathbb{Z}_2) \to F_k(\mathrm{P}^2)$ let us consider the induced commutative diagram which each row is a fibration

$$\begin{tikzcd}
F_k(S^2;\mathbb{Z}_2) \arrow[r] \arrow[d, "\pi"] & ESO(3) \underset{SO(3)}{\times} F_k(S^2;\mathbb{Z}_2) \arrow[r] \arrow[d] & BSO(3) \arrow[d] \\
F_k(\mathrm{P}^2) \arrow[r] & ESO(3) \underset{SO(3)}{\times} F_k(\mathrm{P}^2) \arrow[r] & BSO(3)
\end{tikzcd}$$

Here, $SO(3)$ acts diagonally on $F_k(\mathrm{P}^2)$ considering P^2 as the space of lines through the origin in $\mathbb{R}^3$. Since π is a covering map, it induces an isomorphism in π_i, for all $i \neq 1$. It follows that the vertical map on the middle also induces an isomorphism in π_i, for $i \neq 1$. From the preceding theorem, it follows that $ESO(3) \underset{SO(3)}{\times} F_k(\mathrm{P}^2)$ is a $K(\pi, 1)$. □

Since $1 \times \Sigma_k$ acts freely on $ESO(3) \underset{SO(3)}{\times} F_k(\mathrm{P}^2)$ it follows that

$$ESO(3) \underset{SO(3)}{\times} F_k(\mathrm{P}^2)/\Sigma_k$$

is a $K(\pi, 1)$ with π isomorphic to $\Gamma^k(\mathrm{P}^2)$, by the results on Section 3. This theorem is used in [23] and [10] to get that $ES^3 \times_{S^3} F_k(\mathrm{P}^2)$ and $ES^3 \times_{S^3} F_k(\mathrm{P}^2)/\Sigma_k$ are $K(\pi, 1)$ spaces for the pure braid group $P_k(\mathrm{P}^2)$ and the (full) braid group $B_k(\mathrm{P}^2)$, respectively.

Proposition 4.4. *[23] For $k \geq 2$, the space $ES^3 \times_{S^3} F_k(\mathrm{P}^2)/\Sigma_k$ is a $K(B_k(\mathrm{P}^2), 1)$.*

The proof of this theorem can be obtained from Lemma 3.5 above and certain diagram of fibrations. Moreover, these methods can be applied also for $K(\pi, 1)$ spaces for the braid groups of the 2-sphere S^2.

5 Configurations on the Klein bottle

Let $T^2 = S^1 \times S^1$ be the standard 2-dimensional torus and let $\mathbb{K} = T^2/\sim$ be the Klein bottle, where $(u_1, u_2) \sim (-u_1, \overline{u}_2)$ and $\overline{u}_2$ is complex conjugation. Consider the action

$$\theta : O(2) \times T^2 \longrightarrow T^2,$$

given by $(A, (u_1, u_2)) \mapsto (Au_1, \det(A)u_2)$, where Au_1 is matrix multiplication. Thus, there is a commutative diagram

$$\begin{tikzcd} O(2) \times T^2 \arrow[r, "\theta"] \arrow[d, "1\times\pi"'] & T^2 \arrow[d, "\pi"] \\ O(2) \times \mathbb{K} \arrow[r, "\hat{\theta}"] & \mathbb{K} \end{tikzcd}$$

where $\pi : T^2 \to \mathbb{K}$ is the obvious quotient map, and $\hat{\theta}$ is the induced action on $\mathbb{K}$.

The restriction of the action $\hat{\theta}$ above gives an $SO(2)$-action on $\mathbb{K}$, which is given by matrix multiplication on the first coordinate and leaving the second coordinate fixed:

$$SO(2) \times \mathbb{K} \longrightarrow \mathbb{K}, \quad (A, (u_1, u_2)) \mapsto (Au_1, u_2).$$

Consider the $SO(2)$-action induced on $F_k(\mathbb{K})$ and its associated Borel construction.

Lemma 5.1. *Let $k \geq 1$ and assume that $ESO(2) \underset{SO(2)}{\times} \mathbb{K}$ is a $K(\pi, 1)$ space. Then the space*

$$ESO(2) \underset{SO(2)}{\times} F_k(\mathbb{K})$$

is also a $K(\pi', 1)$.

Remark. The assumption on the $SO(2)$-Borel construction for $\mathbb{K}$ will be proven at the end of this section, so Lemma 5.1 is actually true for $k \geq 1$.

Proof. Consider the Fadell-Neuwirth fibration

$$\mathbb{K}\backslash Q_2 \longrightarrow F_2(\mathbb{K}\backslash Q_1) \longrightarrow \mathbb{K}\backslash Q_1.$$

It follows from the homotopy exact sequence that $F_2(\mathbb{K}\backslash Q_1)$ is a $K(\pi, 1)$. Also, this space is the base space of the fibration

$$\mathbb{K}\backslash Q_3 \longrightarrow F_3(\mathbb{K}\backslash Q_1) \longrightarrow F_2(\mathbb{K}\backslash Q_1)$$

from which one gets that $F_3(\mathbb{K}\backslash Q_1)$ is also a $K(\pi', 1)$. Continuing with this process one can show that $F_k(\mathbb{K}\backslash Q_1)$ is a $K(\pi, 1)$ space. Now consider the fibration induced at the level of Borel constructions

$$F_{k-1}(\mathbb{K}\backslash Q_1) \longrightarrow ESO(2) \underset{SO(2)}{\times} F_k(\mathbb{K}) \longrightarrow ESO(2) \underset{SO(2)}{\times} \mathbb{K}.$$

By hypothesis, the space $ESO(2) \underset{SO(2)}{\times} \mathbb{K}$ is a $K(\pi, 1)$. Then the lemma follows from the associated homotopy exact sequence. $\square$

Corollary 5.2. *For $k \geq 1$, the space $ESO(2) \underset{SO(2)}{\times} F_k(\mathbb{K})/\Sigma_k$ is a $K(\pi, 1)$.*

The rest of the section will be devoted to prove the following resut, which was assumed in Lemma 5.1.

Lemma 5.3. *The space $ESO(2) \underset{SO(2)}{\times} \mathbb{K}$ is a $K(\pi, 1)$.*

Proof. Consider the natural fibration

$$\mathbb{K} \longrightarrow ESO(2) \underset{SO(2)}{\times} \mathbb{K} \longrightarrow BSO(2)$$

and its associated homotopy exact sequence

$$\cdots \longrightarrow \pi_i(\mathbb{K}) \longrightarrow \pi_i(ESO(2) \underset{SO(2)}{\times} \mathbb{K}) \longrightarrow \pi_i(BSO(2)) \longrightarrow \cdots$$

Now, since $\pi_i(\mathbb{K})$ is trivial for $i \geq 2$ and $BSO(2)$ is a $K(\mathbb{Z}, 2)$, the homotopy groups $\pi_i(ESO(2) \underset{SO(2)}{\times} \mathbb{K})$ are trivial for $i \geq 3$. Notice that $\pi_2(ESO(2) \underset{SO(2)}{\times} \mathbb{K})$ is trivial if and only if the boundary map ∂ in the following exact sequence is injective

$$\pi_2(ESO(2) \underset{SO(2)}{\times} \mathbb{K}) \to \pi_2 BSO(2) \xrightarrow{\partial} \pi_1\mathbb{K} \to \pi_1(ESO(2) \underset{SO(2)}{\times} \mathbb{K}).$$

Since $\mathbb{K}$ is a finite dimensional $K(\pi, 1)$ the fundamental group $\pi_1\mathbb{K}$ is torsion-free [2] and thus it suffices to prove that ∂ is nonzero. First consider the diagram

$$\begin{tikzcd}
\pi_2(BSO(2)) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_1(\mathbb{K}) \\
\pi_1\Omega BSO(2) \arrow[d, "\cong"] & \\
\pi_1 SO(2) \arrow[uur, bend right] &
\end{tikzcd}$$

[2] This result can be obtained as a consequence of Proposition 4.2 in [8] on the free resolution induced by the universal cover of a $K(\pi, 1)$ space.

where the isomorphism at the top arises from the homotopy exact sequence for the path-loop fibration of $BSO(2)$ and the isomorphism at the bottom is induced by the natural homotopy equivalence. Via these isomorphisms the boundary map ∂ can be considered as a morphism on fundamental groups $\pi_1 SO(2) \to \pi_1 \mathbb{K}$.

On the other hand, recall there is a $SO(2)$-action on $\mathbb{K}$ by rotation on the first coordinate: (Au_1, u_2). If (u_0, u_0) denotes the base point of $\mathbb{K}$ this action induces a map $\theta : SO(2) \to \mathbb{K}$ given by evaluation of a rotation on (u_0, u_0).

We finish the proof by noting that ∂ is the map induced by θ at the level of fundamental groups

$$\theta_* : \pi_1 SO(2) \longrightarrow \pi_1 \mathbb{K}$$

which is clearly a non-zero homomorphism. □

6 Cohomological considerations

The $K(\pi, 1)$ spaces constructed previously can be used to get a homotopical aproximation to the calculation of the cohomology of mapping class groups and also for braid groups. This is done by the identification $H^*(\pi) = H^*(K(\pi, 1))$, for a commutative ring of coefficients. This approach has been considered in [20], [23], [6] obtaining remarkable information on the cohomological structure. In what follows we will recall some of these results.

The 2-sphere. Let $SO(3)$ acts on S^2 by rotations and consider the associated diagonal action on $F_k(S^2)/\Sigma_k$. The Borel construction for this action

$$ESO(3) \underset{SO(3)}{\times} F_k(S^2)/\Sigma_k$$

is a $K(\pi, 1)$ for the punctured mapping class group $\Gamma^k(S^2)$, for $k \geq 2$, as mentioned in Section 3. Moreover, with mod-2 coefficients this construction gives an isomorphism of $H^*(BSO(3); \mathbf{F}_2)$-modules

$$H^*(\Gamma^{2k}(S^2); \mathbf{F}_2) \cong H^*(BSO(3); \mathbf{F}_2) \otimes H^*(F_{2k}(S^2)/\Sigma_{2k}; \mathbf{F}_2),$$

where the cohomology $H^*(F_{2k}(S^2)/\Sigma_{2k};\mathbf{F}_2)$ can be expressed in terms of the cohomology of braid groups ([9]). Calculations with coefficients in the sign representation $\mathbf{F}(-1)$ and with mod-p coefficients can be obtained by considering certain models of function spaces involving the Borel construction above. See [6].

The projective plane. Recall the $K(\pi,1)$ space obtained in Theorem 4.3 for $\Gamma^k(\mathrm{P}^2)$ is the total space of a fibration with base $BSO(3)$ and fiber $F_k(\mathrm{P}^2)/\Sigma_k$. It turns that its cohomology spectral sequence with coefficients mod-2 collapses at the E_2-term ([20]) and thus there is an isomorphism of modules

$$H^*(\Gamma^k(\mathrm{P}^2);\mathbf{F}_2) \cong H^*(BSO(3);\mathbf{F}_2) \otimes H^*(F_k(\mathrm{P}^2)/\Sigma_k;\mathbf{F}_2)$$

The Klein bottle. The case of the Klein bottle is also treated in [20]. The Borel fibration for the $SO(2)$-action

$$F_k(\mathbb{K})/\Sigma_k \longrightarrow ESO(2) \underset{SO(2)}{\times} F_k(\mathbb{K})/\Sigma_k \longrightarrow BSO(2)$$

has a collapsing spectral sequence on mod-2 cohomology and thus the cohomology of $\Gamma^k(\mathbb{K})$ can be expressed as a direct product of modules:

$$H^*(\Gamma^k(\mathbb{K});\mathbf{F}_2) \cong H^*(BSO(2);\mathbf{F}_2) \otimes H^*(F_k(\mathbb{K})/\Sigma_k;\mathbf{F}_2)$$

The two isomorphisms above express that the mod-2 cohomology of punctured mapping classes is completely determined by the mod-2 cohomology of unordered configuration spaces of surfaces. In [20] this relation is used to obtain information about the cohomology of the groups $\Gamma^k(\mathrm{P}^2)$ and $\Gamma^k(\mathbb{K})$. We recall the method in what follows.

First one considers the homology of $F_k(M)/\Sigma_k$ as part of the homology of a larger space called the *labelled* configuration space $C(M;X)$, which is defined for a CW-complex X with basepoint $*$ as the quotient

$$C(M;X) = \left(\coprod_{j\geq 1} F_j(M) \underset{\Sigma_j}{\times} X^j \right) / \approx,$$

where the relation $\approx$ is generated by

$$(m_1,\ldots,m_j;x_1,\ldots,x_j) \approx (m_1,\ldots,m_{j-1};x_1,\ldots,x_{j-1}), \quad \text{if } x_j = *.$$

This space is filtered by subspaces $C_k(M;X)$ given by all configurations of lenght $\leq k$ and stably splits ([5]) as the wedge of successive quotients $D_k(M;X) = C_k(M;X)/C_{k-1}(M;X)$. Thus there is a isomorphism on reduced homology

$$\widetilde{H}_i C(M;X) \cong \bigoplus_{k=1}^{\infty} \widetilde{H}_i D_k(M;X).$$

Moreover, in the case of $X = S^n$ there is an explicit description of the mod-2 homology of $C(M;S^n)$ as a graded vector space ([7]) in terms of the homology of iterated loop spaces

$$H_*(C(M;S^n)) \cong \bigotimes_{q=0}^{m} H_*(\Omega^{m-q} S^{m+n})^{\otimes \beta_q} \tag{1}$$

where β_q is q-th Betti number of M. Every factor on the right hand is an algebra with weights associated to its generators and the reduced homology of $D_k(M;S^n)$ is the vector subspace generated by all elements of weight k.

On the other hand, it is easy to see ([7]) that the space $D_k(M;S^n)$ is the Thom space of the n-fold sum of the k-dimensional vector bundle

$$\eta : \mathbb{R}^k \underset{\Sigma_k}{\times} F_k(M)/\Sigma_k \longrightarrow F_k(M)/\Sigma_k.$$

and thus there is an isomorphism $H_*(F_k(M)/\Sigma_k) \cong H_{*+kn}(D_k(M;S^n))$, the additive structure of the homology of unordered configurations can be found by counting generators of weight k. The isomorphism (1) above is valid for a general smooth compact manifold M of dimension m and coefficients in a field of characteristic zero. If the coefficients are different to $\mathbf{F}_2$ the proof of the isomorphism (1) requires $m+n$ to be odd.

When M is a surface the mod-2 homology of the labelled configuration space $C(M;S^n)$ is given by the tensor product

$$H_*(\Omega^2 S^{n+2})^{\otimes \beta_0} \otimes H_*(\Omega S^{n+2})^{\otimes \beta_1} \otimes H_*(S^{n+2})^{\otimes \beta_2},$$

where β_0, β_1 and β_2 are the mod-2 Betti numbers of M. More explicitly,

$$H_* C(M;S^n) \cong \mathbf{F}_2[y_0, y_1, \ldots,]^{\otimes \beta_0} \otimes \mathbf{F}_2[x]^{\otimes \beta_1} \otimes (\mathbf{F}_2[u]/u^2)^{\otimes \beta_2},$$

where y_0, x and u are the fundamental classes on degrees $n, n+1$ and $n+2$, respectively, and $y_j = Q_1^j(y_0) = Q_1 Q_1 \cdots Q_1(y_0)$. Here Q_1 is the first Dyer-Lashof operation ([11]). The weights of all generators are given by

$$\omega(u) = 1, \ \ \omega(y_j) = 2^j, \ \ \omega(x^i) = i, \quad \text{for } i = 1, 2, \ldots$$

Thus, for the genus g non-orientable closed surface N_g the isomorphism has the form

$$H_*(C(N_g; S^n)) \cong \ \ \mathbf{F}_2[y_0, y_1, \ldots] \otimes \mathbf{F}_2[x_1, x_2, \ldots, x_g] \otimes \mathbf{F}_2[u]/u^2,$$

and a basis for $\overline{H}_q D_k(N_g; S^n)$ consists of monomials of degree q of the form

$$h = u^e x_1^{a_1} \ldots x_g^{a_g} y_0^{b_0} y_1^{b_1} y_2^{b_2} \ldots y_r^{b_r},$$

for some $r \geq 0, \ e = 0, 1$ and $a_i, b_j \geq 0$, such that

$$\omega(h) = e + \sum_{i=1}^{g} a_i + \sum_{j=0}^{r} 2^j b_j = k$$

For $k = 2$, after counting the degrees of the monomials above one finds that

$$H_q(F_2(N_g)/\Sigma_2; \mathbf{F}_2) = \begin{cases} \mathbf{F}_2 & \text{if } \ q = 0 \\ \mathbf{F}_2^{g+1} & \text{if } \ q = 1 \\ \mathbf{F}_2^{g^2+1} & \text{if } \ q = 2 \\ \mathbf{F}_2^{g} & \text{if } \ q = 3 \\ 0 & \text{otherwise} \end{cases}$$

For $k = 2$ and $g = 1$, one has

$$H_q(F_2(\mathrm{P}^2)/\Sigma_2; \mathbf{F}_2) = \begin{cases} \mathbf{F}_2 & \text{if } \ q = 0 \\ \mathbf{F}_2^2 & \text{if } \ q = 1 \\ \mathbf{F}_2^2 & \text{if } \ q = 2 \\ \mathbf{F}_2 & \text{if } \ q = 3 \\ 0 & \text{otherwise} \end{cases}$$

This can be compared with the calculations in [15] where the ring structure of $H^*(F_2(\mathrm{P}^2)/\Sigma_2)$ was obtained motivated by the symmetric topological complexity of real projective spaces P^{n} and the problem of embeddings $\mathrm{P}^{\mathrm{n}} \subset \mathbb{R}^{\mathrm{m}}$. The cohomology of $F_2(\mathrm{P}^2)/\Sigma_2$ is a truncated ring polynomial on two 1-dimensional variables, given in terms of the generator $z \in H^1(\mathrm{P}^\infty)$.

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