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Uniqueness of the Solution of the Yule-Walker Equations: A Vector Space Approach *

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Abstract

This work concerns the Yule-Walker system of linear equations arising in the study of autoregressive processes. Given a complex polynomial $\varphi(z)$ satisfying $\varphi(0) = 1$, elementary vector space ideas are used to derive an explicit formula for the determinant of the matrix $M(\varphi)$ of the Yule-Walker system of equations corresponding to φ. The main conclusion renders the following non-singularity criterion: The matrix $M(\varphi)$ is invertible if and only if the product of two roots of φ is always different form 1, a property that yields that the Yule-Walker system associated with a causal polynomial has a unique solution. The way in which this result is implicitly used in the time series literature is briefly discussed.

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1 Introduction

This note concerns a basic question involving the Yule-Walker equations in time series analysis. To describe the problem we are interested in,

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let $\{X_t\}$ be a zero-mean (second order) stationary process which is supposed to be real-valued and autoregressive of order p ($AR(p)$), that is, $\{X_t\}$ satisfies a difference equation of the form

$$X_t + \varphi_1 X_{t-1} + \cdots + \varphi_p X_{t-p} = Z_t, \tag{1}$$

where the Z_t's are zero-mean uncorrelated random variables with common variance $\sigma^2 > 0$; see, for instance, Anderson (1971) pp. 166–176, or Box and Jenkins (1976), pp. 53–65. It is known that a process $\{X_t\}$ satisfying (1) exists if and only if the autoregressive polynomial $\varphi(z) := 1 + \varphi_1 z + \cdots + \varphi_p z^p$ is such that $\varphi(z) \neq 0$ for all complex z with $|z| = 1$, and in this case, there is not loss of generality in assuming that the polynomial $\varphi(z)$ is causal, i.e., $\varphi(z) \neq 0$ for all z with $|z| \leq 1$, a condition that is supposed to hold in the following discussion; for details, see Remarks 3 and 5 in Brockwell and Davis (1987), pp. 86–88. Now, consider the problem of determining the autocovariance function $\gamma(\cdot)$ of $\{X_t\}$, which is given by $\gamma(h) := \operatorname{Cov}(X_{t+h}, X_t)$, $h = 0, \pm 1, \ldots$. Multiplying both sides of (1) by X_{t-i} and taking the expectation in both sides of the resulting equality, it follows that $\sum_{k=0}^{p} \gamma(|i-k|)\varphi_k = 0$ for $i > 0$, and $\sum_{k=0}^{p} \gamma(k)\varphi_k = \sigma^2$ if $i = 0$, where $\varphi_0 = \varphi(0) = 1$. The following two-step method, which is described in Brockwell and Davis (1987), p. 97, is a computationally convenient tool to determine $\gamma(\cdot)$.

Step 1. Find $\gamma(0), \gamma(1), \ldots, \gamma(p)$ by solving

$$\begin{aligned} \sum_{k=0}^{p} \gamma(k)\varphi_k &= \sigma^2 \\ \sum_{k=0}^{p} \gamma(|i-k|)\varphi_k &= 0, \end{aligned} \tag{2}$$

which is the Yule-Walker (Y-W) system of equations associated to $\varphi(\cdot)$.

Step 2. Using that $\gamma(i) = -\sum_{k=1}^{p} \gamma(i-k)\varphi_k$ for $i > p$, determine $\gamma(p+1), \gamma(p+2), \ldots$ in a recursive way.

In order to have that the above method relies on firm grounds, it is necessary to show that (2) has a unique solution, a fact that can be easily verified when the degree of $\varphi(z)$ is small, say $p = 1$ or $p = 2$; see Section 3 below. For a polynomial of arbitrary degree, Achilles (1987) used an argument based on matrix theory to obtain a formula for the determinant of matrix $M(\varphi)$ of the above Y-W linear system associated to a polynomial φ, and a more compact and advanced approach was used in Lütkepohl and Maschke (1988). On the other hand, it is interesting to observe that the Yule–Walker equations are used (i) to find moment estimators of the polynomial φ (see, for instance, Section 8.2 in Brockwell and Davis, 1991) and (ii) To compute recursively the covariance function of a general ARMA(p, q) process (Brockewll and Davis, 1991, Cavazos-Cadena, 1994). Also, recently an efficient way to implement the innovations algorithm to construct best linear predictors *via* the Durbin-Levinson algorithm (using the non-singularity of $M(\varphi)$ for a causal polynomial φ) was obtained in Martínez–Martínez (2010).

The *main objective* of this work is to present *an elementary derivation of the determinant of the matrix $M(\varphi)$ using simple vector spaces ideas*. The result in this direction, presented in Theorem 3.1 below, yields the following criterion: The Y-W system associated with the polynomial $\varphi(\cdot)$ has a unique solution if and only if

$$r_i r_j \neq 1, \quad i, j = 1, 2, \ldots, p, \tag{3}$$

where $r_1, \ldots, r_p$ are the roots of $\varphi(z)$. Notice that when $\varphi(z)$ is a causal polynomial all of its roots r_i lie outside the unit disk, and in this case (3) is clearly satisfied.

The proof of Theorem 3.1 is based on an induction argument using standard ideas on vector spaces, and the exposition has been organized as follows: In Section 2 the basic (infinite–dimensional) vector space is introduced, and the main result on the determinant of the matrix $M(\varphi)$ is stated in Section 3. Next, in Section 4 some necessary preliminaries to prove Theorem 3.1 are presented, and the proof of the main result is given in Section 5. Finally, the presentation concludes with some brief comments in Section 6.

2 Auxiliary Vector Space and Basic Notation

Throughout the remainder $\mathbf{Z}$ and $\mathbb{N}$ stand for the sets of all integers and all nonnegative integers, respectively, and $\mathcal{C}$ denotes the set of all complex numbers. The complex vector space $\mathcal{L}$ consists of all vectors $\mathbf{v} : \mathbb{N} \to \mathcal{C}$ with the property that $\mathbf{v}(k) = 0$ for all k large enough, and $\mathcal{L}$ is endowed with the usual addition and scalar multiplication. The shift operator $s\colon \mathcal{L} \to \mathcal{L}$ is defined as follows: For $\mathbf{v} \in \mathcal{L}$

$$s(\mathbf{v})(0) := 0, \quad \text{and} \quad s(\mathbf{v})(k) := \mathbf{v}(k-1), \quad k = 1, 2, \ldots; \tag{4}$$

in addition, the n-fold composition of the shift operator with itself is denoted by s^n, so that

$$s^0(\mathbf{v}) = \mathbf{v}, \quad \text{and} \quad s^n(\mathbf{v}) = s^{n-1}(s(\mathbf{v})). \tag{5}$$

On the other hand, rows and columns of a squared matrix M are numbered starting from zero, and Det M denotes the determinant of M. For vectors $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{L}$, the corresponding squared matrix $M_{n+1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n)$ of order (n + 1) is defined by

$$M_{n+1}(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n) := \mathbf{v}_i(j), \quad i, j = 0, 1, \ldots, n, \tag{6}$$

whereas span$\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ stands for the vector space generated by the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_n$; the corresponding dimension is denoted by

$$\dim \operatorname{span}\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}.$$

To conclude, with a given a polynomial $\varphi(z) = \varphi_0 + \varphi_1 z + \cdots + \varphi_p z^p$ of degree p, the vectors $\overrightarrow{\varphi}$, $\overleftarrow{\varphi} \in \mathcal{L}$ are defined as follows:

$$\begin{array}{ll} \text{For } k = 0, 1, \ldots, p & \overrightarrow{\varphi}(k) := \varphi_k \text{ and } \overleftarrow{\varphi}(k) := \varphi_{p-k}, \\ \text{and, for } k > p, & \overrightarrow{\varphi}(k) = \overleftarrow{\varphi}(k) = 0. \end{array} \tag{7}$$

Finally, the following notational convention concerning the coefficients of the polynomial $\varphi(z)$ will be used:

$$\varphi_k := 0 \quad \text{for } k < 0 \text{ or } k > p. \tag{8}$$

3 Main Result

Let $\varphi(z)$ be a polynomial of degree p. The objective of this section is to state a formula for the determinant of the matrix corresponding to the Y-W system (2). To begin with, notice that for $i > 0$,

$$\begin{aligned} \sum_{k=0}^{p} \gamma(|i-k|)\varphi_k &= \sum_{k=0}^{i} \gamma(i-k)\varphi_k + \sum_{k=i+1}^{p} \gamma(k-i)\varphi_k \\ &= \sum_{j=0}^{i} \gamma(j)\varphi_{i-j} + \sum_{j=1}^{p-i} \gamma(j)\varphi_{i+j} \end{aligned}$$

and using convention (8) it follows that

$$\sum_{k=0}^{p} \gamma(|i-k|)\varphi_k = \gamma(0)\varphi_i + \sum_{j=1}^{p} \gamma(j)[\varphi_{i-j} + \varphi_{i+j}].$$

Thus, the Y-W system (2) can be equivalently written as

$$\begin{aligned} \sum_{k=0}^{p} \gamma(k)\varphi_k &= \sigma^2 \\ \gamma(0)\varphi_i + \sum_{j=1}^{p} \gamma(j)[\varphi_{i-j} + \varphi_{i+j}] &= 0, \quad i = 1, 2, \ldots, p. \end{aligned} \tag{9}$$

The (squared) matrix of this system will be denoted by $M(\varphi)$. Clearly, $M(\varphi)$ is of order $(p+1)$ and is given as follows: For $i = 0, 1, 2, \ldots, p$,

$$M(\varphi)_{i\,0} := \varphi_i, \text{ and } M(\varphi)_{i\,j} := \varphi_{i-j} + \varphi_{i+j}, \quad j = 1, 2, \ldots, p; \tag{10}$$

for instance, $M(\varphi)_{0\,0} = \varphi_0$ and, for $j > 0$, $M(\varphi)_{0\,j} = \varphi_{0-j} + \varphi_{0+j} = \varphi_j$, in accordance with the first equation in (9). The next theorem contains a formula for the determinant of $M(\varphi)$ and, as a by-product, a criterion for the non-singularity of $M(\varphi)$ is obtained.

Theorem 3.1. *Let $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ be a complex polynomial of degree p. If the roots of $\varphi(\cdot)$ are $r_1, \ldots, r_p$ and $M(\varphi)$ is as in (10), then assertions (i) and (ii) below occur.*

(i) The determinant of $M(\varphi)$ is given by

$$\text{Det } M(\varphi) = \prod_{1 \le i < j \le p} [1 - (r_i r_j)^{-1}] \prod_{i=1}^{p} (1 - r_i^{-2}) \tag{11}$$

where, by (the usual) convention, for $p = 1$ the first product in the above display is 1.

Consequently,

(ii) $M(\varphi)$ is non-singular if and only if $r_i r_j \neq 1$ for all $i, j = 1, \ldots, p$.

This result will be established in Section 5; by the moment, it is convenient to note that (ii) follows immediately from part (i). On the other hand, (11) is easily verified for small values of p. For instance, for $p = 1$, the polynomial $\varphi(z)$ is given by $\varphi(z) = 1 + \varphi_1 z$ and (10) yields that

$$M(\varphi) = \begin{bmatrix} 1 & \varphi_1 \\ \varphi_1 & 1 \end{bmatrix}$$

so that Det $M(\varphi) = 1 - \varphi_1^2$, and this yields (11) with $p = 1$, since $\varphi(\cdot)$ has the unique root $r_1 = -1/\varphi_1$. For $p = 2$ factorize $\varphi(z)$ as

$$\varphi(z) = (1 + a_1 z)(1 + a_2 z),$$

where the roots of $\varphi(z)$ are $r_i = -1/a_i$, $i = 1, 2$. It follows that

$$\varphi(z) = 1 + (a_1 + a_2)z + a_1 a_2 z^2$$

and $M(\varphi)$ is given by

$$M(\varphi) = \begin{bmatrix} 1 & a_1 + a_2 & a_1 a_2 \\ a_1 + a_2 & a_1 a_2 + 1 & 0 \\ a_1 a_2 & a_1 + a_2 & 1 \end{bmatrix}$$

Then, expanding $Det M(\varphi)$ by the third column the following expression is obtained:

$$\begin{aligned} \text{Det } M(\varphi) &= a_1 a_2[(a_1 + a_2) - a_1 a_2(1 + a_1 a_2)] \\ &\quad +(1 + a_1 a_2) - (a_1 + a_2)^2 \\ &= -(a_1 + a_2)^2(1 - a_1 a_2) + (1 + a_1 a_2)[1 - (a_1 a_2)^2] \\ &= (1 - a_1 a_2)[-(a_1 + a_2)^2 + (1 + a_1 a_2)^2] \\ &= (1 - a_1 a_2)(1 - a_1^2)(1 - a_2^2) \end{aligned}$$

and replacing a_i by $-1/r_i$ the formula in (11) is obtained for the case $p = 2$. The proof of (11) in the general case is by induction and is presented in Section 5.

4 Preliminaries

This section contains the technical tools that will be used to establish Theorem 3.1. The starting point is the idea in the following definition.

Definition 4.1. . *Let $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ be a polynomial of degree p. The sequence $\mathbf{V}^\varphi = \{V_t^\varphi \,|\, t \in \mathbf{Z}\} \subset \mathcal{L}$ is defined as follows: (i) For $0 \le n < p$,*

$$V_n^\varphi(0) := \varphi_n, \quad \textit{and} \quad V_n^\varphi(k) := \varphi_{n-k} + \varphi_{n+k}, \quad k = 1, 2, \ldots.$$

(ii) For $n \in \mathbb{N}$

$$V_{-n}^\varphi := s^n(\overrightarrow{\varphi}) \quad \textit{and} \quad V_{n+p}^\varphi := s^n(\overleftarrow{\varphi});$$

see (4)–(7) for notation.

A glance to the above definition and (4)–(8) shows that the sequence $\mathbf{V}^\varphi$ is related to the matrix $M(\varphi)$ through the following equality:

$$M(\varphi) = M_{p+1}(V_0^\varphi, V_1^\varphi, \ldots, V_p^\varphi). \tag{12}$$

The following is the key technical result of this section.

Theorem 4.2. *Suppose that $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ has degree p and satisfies $\varphi(b) = \varphi(1/b) = 0$ for some $b \in \mathcal{C} \setminus \{0\}$. In this case the following assertions (i)–(iii) occur.*

(i) $\dim \operatorname{span}\{V_{-1}^\varphi, V_0^\varphi, \ldots, V_{p+1}^\varphi\} \le p + 1$.

(ii) For each $a \in \mathcal{C}$,

$$\operatorname{Det}\, M_{p+2}(V_0^\varphi + aV_{-1}^\varphi, V_1^\varphi + aV_0^\varphi, \ldots, V_{p+1}^\varphi + aV_p^\varphi) = 0.$$

(iii) For all $a \in \mathcal{C}$, $\operatorname{Det}\, M[(1 + az)\varphi(z)] = 0$.

The proof of Theorem 4.1 has been divided into several pieces presented in the Lemmas 4.1–4.4 below, which involve the notion introduced below.

Definition 4.3. *Let $\mathbf{V} = \{V_t | t \in \mathbf{Z}\}$ be a sequence in $\mathcal{L}$.*

(i) $\mathbf{V}$ has property $\mathcal{D}(p)$ if for all $n \in \mathbb{N}$

$$\dim \operatorname{span}\{V_t | -n \le t \le p + n\} \le p + n.$$

(ii) Given $a \in \mathcal{C}$, the sequence $T_a\mathbf{V} = \{T_a V_t \,|\, t \in \mathbf{Z}\}$ is defined by

$$[T_a\mathbf{V}]_t \equiv T_a V_t := V_t + aV_{t-1}, \quad t \in \mathbf{Z}.$$

The starting point of the journey to the proof of Theorem 4.2 is the following.

Lemma 4.4. *Let $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ be a polynomial of degree p and $a \in \mathcal{C} \setminus \{0\}$. If $\theta(z) = (1 + az)\varphi(z)$, then $\mathbf{V}^\theta = T_a\mathbf{V}^\varphi$*

Proof. Let $n \in \{1, 2, \ldots, p\}$ be arbitrary. From Definition 4.1 it follows that

$$V_n^\theta(0) = \theta_n = \varphi_n + a\varphi_{n-1} = V_n^\varphi(0) + aV_{n-1}^\varphi(0) = T_a V_n^\varphi(0),$$

and for $k = 1, 2, \ldots,$

$$\begin{aligned} V_n^\theta(k) &= \theta_{n+k} + \theta_{n-k} \\ &= (\varphi_{n+k} + a\varphi_{n-k-1}) + (\varphi_{n-k} + a\varphi_{n-k-1}) \\ &= (\varphi_{n+k} + \varphi_{n-k}) + a[\varphi_{n-1-k} + \varphi_{n-1+k}] \\ &= V_n^\varphi(k) + aV_{n-1}^\varphi(k) \\ &= T_a V_n \varphi(k) \end{aligned}$$

These two last displays show that, for $1 \le n \le p$, $V_n^\theta = T_a V_n^\varphi$, and to complete the proof this equality should be verified for $n < 0$ and $n > p$. To achieve this goal, first notice that $\overrightarrow{\theta} = \overrightarrow{\varphi} + as(\overrightarrow{\varphi})$ and $\overleftarrow{\theta} = s(\overleftarrow{\varphi}) + a\overleftarrow{\varphi}$, a fact that can be obtained from (4) and (7). Then, for $n \ge 0$,

$$\begin{aligned} V_{-n}^\theta &= s^n(\overrightarrow{\theta}) \\ &= s^n[\overrightarrow{\varphi} + as(\overrightarrow{\varphi})] \\ &= s^n(\overrightarrow{\varphi}) + as^{n+1}(\overrightarrow{\varphi}) \\ &= V_{-n}^\varphi + aV_{-n-1}^\varphi \\ &= T_a V_{-n}^\varphi \end{aligned}$$

Similarly,

$$\begin{aligned} V_{n+p+1}^\theta &= s^n(\overleftarrow{\theta}) \\ &= s^n[s(\overleftarrow{\varphi}) + a\overleftarrow{\varphi}] \\ &= s^{n+1}(\overleftarrow{\varphi}) + as^n(\overleftarrow{\varphi}) \\ &= V_{n+1+p}^\varphi + aV_{n+p}^\varphi \\ &= T_a V_{n+1+p}^\varphi. \end{aligned}$$

Thus, it has been established that $V_{-n}^\theta = T_a V_{-n}^\varphi$ and $V_{n+1+p}^\theta = T_a V_{n+1+p}^\varphi$ for all $n \in \mathbb{N}$; as already noted, this completes the proof. □

The following lemma studies the relation of property $\mathcal{D}(k)$ with the transformation T_a; see Definition 4.3.

Lemma 4.5. *Let $a \in \mathcal{C}$ be arbitrary and suppose that $\mathbf{V} = \{V_t \,|\, t \in \mathbf{Z}\} \subset \mathcal{L}$ has property $\mathcal{D}(k)$. Then $T_a\mathbf{V}$ has property $\mathcal{D}(k+1)$*

Proof. Notice that $T_a V_t \in \text{span}\{V_t, V_{t-1}\}$ and then, for arbitrary $r \in \mathbb{N}$, $\text{span}\{T_a V_t | -r \le t \le k+1+r\} \subset \text{span}\{V_t | -(r+1) \le t \le k+(r+1)\}$.

Now observe that, since the sequence $\mathbf{V}$ has the property $\mathcal{D}(k)$, the space in the right-hand side of the previous equality has dimension less that or equal to $k+r+1$ and, consequently,

$$\dim \operatorname{span}\{T_a V_t \,|\, -r \leq t \leq (k+1)+r\} \leq (k+1)+r,$$

a relation that shows that $T_a\mathbf{V}$ has property $\mathcal{D}(k+1)$, since r was arbitrary. □

The next two lemmas relate property $\mathcal{D}(k)$ with sequences of the form $\mathbf{V}^\varphi$; see Definition 4.1.

Lemma 4.6. *Let $\varphi(z)$ be a polynomial of degree $p \geq 1$ and suppose that $\varphi(1) = 0$ or $\varphi(-1) = 0$. In this case $\mathbf{V}^\varphi$ has property $\mathcal{D}(p)$.*

Proof. First suppose that $p = 1$. When $\varphi(1) = 0$ the polynomial $\varphi(z)$ is given by $\varphi(z) = \varphi(0)(1-z)$ and, using (8), it follows that $\overrightarrow{\varphi} = -\overleftarrow{\varphi}$, which yields that, for $n \geq 0$, $V^\varphi_{-n} = s^n(\overrightarrow{\varphi}) = -s^n(\overleftarrow{\varphi}) = -V^\varphi_{n+1}$. If $\varphi(-1) = 0$ it follows that $\varphi(z) = \varphi(0)(1+z)$ and then $\overrightarrow{\varphi} = \overleftarrow{\varphi}$, and then $V^\varphi_{-n} = V^\varphi_{n+1}$, $n \in \mathbb{N}$. Therefore, in either case, for every $r \in \mathbb{N}$,

$$\operatorname{span}\{V^\varphi_t \,|\, -r \leq t \leq 1+r\} = \operatorname{span}\{V^\varphi_t \,|\, 1 \leq t \leq 1+r\}$$

and since the space in the right-hand side has $r+1$ generators, it follows that $\dim \operatorname{span}\{V^\varphi_t \,|\, -r \leq t \leq 1+r\} \leq r+1$, *i.e*, $\mathbf{V}^\varphi$ has property $\mathcal{D}(1)$; see Definition 4.1. The proof will be now completed by induction in p. Suppose that the result holds for $p = k$, and let φ be a polynomial of degree $k+1$ vanishing at 1 or -1. In this case it is clearly possible to factorize $\varphi(z)$ as $\varphi(z) = (1+az)\theta(z)$, where $a \in \mathcal{C}$ and $\theta(z)$ has degree k and satisfies $\theta(1) = 0$ or $\theta(-1) = 0$. By the induction hypothesis, $\mathbf{V}^\theta$ has property $\mathcal{D}(k)$ and, using Lemmas 4.4 and 4.5, it follows that $\mathbf{V}^\varphi = T_a\mathbf{V}^\theta$ has the property $\mathcal{D}(k+1)$. □

The following is the final step before the proof of Theorem 4.1.

Lemma 4.7. *Let $\varphi(z)$ be a polynomial of degree $p \geq 2$ such that $\varphi(b) = \varphi(1/b) = 0$ for some $b \in \mathcal{C} \setminus \{0, 1, -1\}$. Then, $\mathbf{V}^\varphi$ has the property $\mathcal{D}(p)$*

Proof. The argument is along the same lines as in the proof of Lemma 4.6. First suppose that $p = 2$. In this case

$$\varphi(z) = \varphi(0)(1-bz)(1-z/b) = \varphi(0)[1-(b+b^{-1})z+z^2],$$

and using (7) it follows that $\overrightarrow{\varphi} = \overleftarrow{\varphi}$; by Definition 4.1 (i) this yields that $V^\varphi_{-n} = V^\varphi_{1+n}$ for every $n \geq 0$, and then, for each $r \in \mathbb{N}$,

$$\mathrm{span}\{V^\varphi_t \,|\, -r \leq t \leq 2+r\} = \mathrm{span}\{V^\varphi_t \,|\, 1 \leq t \leq 2+r\},$$

and since the vector space in the right-hand side has $r+2$ generators this yields that

$$\dim \mathrm{span}\{V^\varphi_t \,|\, -r \leq t \leq 2+r\} \leq r+2,$$

that is, $\mathbf{V}^\varphi$ has the property $\mathcal{D}(2)$. The result for arbitrary p is obtained by an induction argument similar to that used in the proof of Lemma 4.6. □

The previous lemmas are used below to establish the main result of this section.

Proof of Theorem 4.2. Let $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ be a polynomial of degree p with $\varphi(b) = \varphi(1/b) = 0$ for some $b \in \mathcal{C} \setminus \{0\}$

(i) When $b = 1$ or $b = -1$ Lemma 4.6 yields that $\mathbf{V}^\varphi$ has property $\mathcal{D}(p)$, and Lemma 4.7 implies that the same conclusion holds when $b \neq 1, -1$. Then, by Definition 4.3(i), it follows that

$$\dim \mathrm{span}\{V^\varphi_{-1}, V^\varphi_0, \ldots, V^\varphi_p, V^\varphi_{p+1}\} \leq p+1.$$

(ii) Notice that

$$\mathrm{span}\{V^\varphi_r + aV^\varphi_{r-1} \,|\, r = 0, 1, \ldots, p+1\} \subset \mathrm{span}\{V^\varphi_t \,|\, -1 \leq t \leq p+1\},$$

and then $\dim \mathrm{span}\{V^\varphi_r + aV^\varphi_{r-1} \,|\, r = 0, 1, \ldots, p+1\} \leq p+1$, by part (i). It follows that the $p+2$ vectors $V^\varphi_r + aV^\varphi_{r-1}$, $r = 0, 1, 2, \ldots, (p+1)$ are linearly dependent in $\mathcal{L}$, a fact that implies the linear dependence of the rows of $M_{p+2}(V^\varphi_0 + aV^\varphi_{-1}, \ldots, V^\varphi_{p+1} + aV^\varphi_p)$; as a consequence,

$$\mathrm{Det}\; M_{p+2}(V^\varphi_0 + aV^\varphi_{-1}, \ldots, V^\varphi_{p+1} + aV^\varphi_p) = 0;$$

see, for instance, Chapter 5 of Hoffman and Kunze (1971).

(iii) Set $\psi(z) := (1 + az)\varphi(z)$. In this case $\psi(z)$ has degree $p+1$, and using (12) with $p+1$ and ψ instead of p and φ, respectively, it follows that that

$$\begin{aligned} M(\psi) &= M_{p+2}(V^\psi_0, V^\psi_1, \ldots, V^\psi_{p+1}) \\ &= M_{p+2}(V^\varphi_0 + aV^\varphi_{-1}, \ldots, V^\varphi_{p+1} + aV^\varphi_p) \end{aligned}$$

where Lemma 4.4 was used to set the second equality; from this point, the previous part yields that Det $M(\psi) = 0$. □

This section concludes with a simple a simple fact that will be useful in the proof of Theorem 3.1.

Lemma 4.8. *Let $\varphi(z)$ be a polynomial of degree p with $\varphi(0) = 1$. In this case*

$$\text{Det } M(\varphi) = \text{Det } M_{p+2}(V_0^\varphi, V_1^\varphi, \ldots, V_{p+1}^\varphi).$$

Proof. By convenience set $L := M_{p+2}(V_0^\varphi, V_1^\varphi, \ldots, V_{p+1}^\varphi)$ and observe the following facts (a) and (b).

(a) A glance to (6) and Definition 4.1(i) shows that the submatrix obtained by deleting the last row and the last column of L is

$$M_{p+1}(V_0^\varphi, V_1^\varphi, \ldots, V_p^\varphi).$$

Next, the components in the last column of L will be evaluated. First recall Definition 4.2(i) and notice that $L_{0\,p+1} = V_0^\varphi(p+1) = s^0(\overrightarrow{\varphi})(p+1) = \overrightarrow{\varphi}(p+1) = 0$; see (5) and (7). On the other hand, for $1 \leq n < p$, $L_{n\,p+1} = \varphi_{n+p+1} + \varphi_{n-p-1} = 0$, where convention 8) was used to set the last equality. Finally, $L_{p\,p+1} = V_p^\varphi(p+1) = s^0(\overleftarrow{\varphi})(p+1) = \overleftarrow{\varphi}(p+1) = 0$, and $L_{p+1\,p+1} = s(\overleftarrow{\varphi})(p+1) = \overleftarrow{\varphi}(p) = \varphi_0 = \varphi(0) = 1$. Summarizing:

(b) The last column of L consists entirely of zeros except by the element in the last row, which is one. To conclude, expand Det L across the last column. In this case the facts (a) and (b) above together imply that Det $L = M_{p+1}(V_0^\varphi, V_1^\varphi, \ldots, V_p^\varphi) =$ Det $M(\varphi)$, where the last equality follows from (12). □

5 Proof of Theorem 3.1

The preliminary results in the previous section will be used to establish the main result of this note.

Proof of Theorem 3.1. As already mentioned it is sufficient to prove part (i). Let $\varphi(z) = 1 + \varphi_1 z + \cdots + \varphi_p z^p$ be a polynomial of degree p and factorize φ as

$$\varphi(z) = \prod_{i=1}^{p}(1 + a_i z),$$

where the roots of $\varphi(z)$ are $-1/a_i$, $i = 1, 2, \ldots, p$. With this notation (11) is equivalent to

$$\text{Det } M\left[\prod_{i=1}^{p}(1 + a_i z)\right] = \prod_{l \le i < j \le p}[1 - a_i a_j]\prod_{i=1}^{p}(1 - a_i^2),$$

an equality that was verified in Section 3 for $p = 1$ and $p = 2$. The proof of (5) will be completed by induction. Suppose that (5) holds for $p = n \ge 2$ and let $a_1, a_2, \ldots, a_{n+1}$ be no-null complex numbers. Define

$$\psi(z) := \prod_{i=1}^{n}(1 + a_i z)$$

and, for each $c \in \mathcal{C}$, set

$$F(c) := \text{Det } M_{n+2}(V_0^{\psi} + cV_{-1}^{\psi}, \ldots, V_{n+1}^{\psi} + cV_n^{\psi}).$$

Combining Lemma 4.4 and (12) it follows that

(a) $F(c) = \text{Det } M[(1 + cz)\psi(z)]$; in particular,

$$F(a_{n+1}) = \text{Det } M\left[\prod_{i=1}^{n+1}(1 + a_i z)\right].$$

Using the multilinearity of the determinant function, (5) yields that

(b) $F(c)$ is a polynomial in c with degree $\le n + 2$; see, for instance, Hoffmann and Kunze (1971) Chapter 5.

Next, the roots of the polynomial $F(c)$ will be determined. First observe that

(c) $F(1) = F(-1) = 0$.

To verify this last assertion set $\psi^*(z) := (1 + z)\prod_{i=2}^{p}(1 + a_i z)$ and notice that $\psi^*(-1) = 0$, and $(1+z)\psi(z) = (1+a_1 z)\psi^*(z)$; see (5). Then the above property (a) yields that

$$F(1) = \text{Det } M[(1 + z)\psi(z)] = \text{Det } M[(1 + a_1 z)\psi^*(z)] = 0,$$

where the last equality is due to Theorem 4.2(iii) with $\psi^*(z)$ and -1 instead of $\varphi(z)$ and b, respectively. Similarly, it can be shown that $F(-1) = 0$.

(d) $F(1/a_i) = 0$ for $i = 1, 2, \ldots, n$. To show this, let $k, i \in \{1, 2, \ldots, n\}$

be fixed integers with $k \neq i$ and define

$$\tilde{\psi}(z) := (1 + z/a_i) \prod^{(k)} (1 + a_j z),$$

where $\prod^{(k)}$ indicates the product over all the integers j between 1 and n satisfying $j \neq k$; notice that

$$(1 + z/a_i)\psi(z) = (1 + a_k z)\tilde{\psi}(z). \tag{13}$$

From (5) it is clear that $\tilde{\psi}(-a_i) = 0$ and, since $k \neq i$, the polynomial $\tilde{\psi}(z)$ contains the factor $(1 + a_i z)$, and then $\tilde{\psi}(-1/a_i) = 0$. Therefore, from (a) and (13) it follows that $F(1/a_i) = \text{Det } M[(1 + z/a_i)\psi(z)] = \text{Det } M[(1 + a_k z)\tilde{\psi}(z)]$, and an application of Theorem 4.1(iii) with $\tilde{\psi}$ and $-a_i$ instead of φ and b, respectively, yields that $F(1/a_i) = 0$.

To continue, suppose by the moment that $a_1, a_2, \ldots, a_n$ are different numbers in $\mathcal{C} \setminus \{0, 1, -1\}$. In this case, the above facts (c) and (d) together show that the polynomial $F(c)$ has $n+2$ different roots, namely, $1, -1$, and $1/a_i$, $i = 1, 2, \ldots, n$. Combining this fact with (b) it follows that $F(\cdot)$ has degree $n + 2$ and it can be factorized as

$$F(c) = F(0)(1 - c)(1 + c) \prod_{i=1}^{n} (1 - a_i c).$$

Setting $c = a_{n+1}$ and using (a) it follows that

$$\text{Det } M \left[\prod_{i=1}^{n+1} (1 + a_i z) \right] = F(0)(1 - a_{n+1}^2) \prod_{i=1}^{n} (1 - a_i a_{n+1}).$$

Next, using (5) it follows that $F(0) = \text{Det } M_{n+2}(V_0^\psi, V_1^\psi, \ldots, V_{n+1}^\psi)$, and then $F(0) = M(\psi)$, by Lemma 4.8 applied to ψ, which has degree n, and then the induction hypothesis yields that

$$F(0) = \prod_{i=1}^{n} (1 - a_i^2) \prod_{1 \leq i < j \leq n} (1 - a_i a_j),$$

and combining this equality with (5) it follows that

$$\text{Det } M \left[\prod_{i=1}^{n+1} (1 + a_i z) \right] = \prod_{i=1}^{n+1} (1 - a_i^2) \prod_{1 \leq i < j \leq n+1} (1 - a_i a_j), \tag{14}$$

which is (5) with $p = n + 1$. Although (14) has been established under the assumption that $a_1, \ldots, a_{n+1}$ are different numbers in $\mathcal{C} \setminus \{0, 1, -1\}$, the equality holds for arbitrary $a_i, \ldots, a_{n+1} \in \mathcal{C} \setminus \{0\}$, since both sides of (14) are continuous functions of the a_is. In short, assuming that (5) holds for $p = n$ it has been shown that it is also valid for $p = n + 1$ completing the induction argument.

6 Concluding Remarks

Given a polynomial $\varphi(z)$ with $\varphi(0) = 1$, a necessary and sufficient condition was established so that the corresponding Yule-Walker system has a unique solution, and it has been shown that such a condition is satisfied by a causal polynomial. Besides to provide a rigorous basis for the two step method described in Section 1, there are other parts in the theory of time series where it is important to know that a Yule-Walker system has a unique solution. For instance, consider the following result: Given p >0 and an autocovariance function $\gamma(\cdot)$ with $\gamma(h) \to 0$ as $h \to \infty$ there exists an $AR(p)$ process $\{Y_t\}$ whose autocovariance function $\gamma_Y(\cdot)$ coincides with $\gamma(\cdot)$ at lags $h = 0, 1, \ldots, p$. A proof of this fact can be found in Brockwell and Davis (1987), pp. 232-233, and here we just mention that there is a passage where, for a certain causal polynomial $\varphi(\cdot)$, it is shown that both $\{\gamma(h)|0 \leq h \leq p\}$ and $\{\gamma(h) \,|\, 0 \leq h \leq p\}$ satisfy (2), and then it is immediately concluded that $\gamma(h) = \gamma_Y(h)$ for $0 \leq h \leq p$. Thus, it is implicitly assumed that the Yule-Walker system of a causal polynomial has a unique solution, a fact that, by Theorem 3.1, is indeed true.

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