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The minimum cost flow problem with interval and fuzzy arc costs *

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Abstract

We follow the total order for intervals and fuzzy numbers introduced by Hashemi et al. in [1] and Ghatee et al. in [2] to solve the minimum cost flow problem with either, interval or fuzzy arc costs by using its crisp model, a minimum cost flow problem associated to original imprecise problem. Numerical simulations compare the performance of this method in real scenarios with the algorithm proposed in [1].

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1 Introduction

The Minimum Cost Flow Problem (MCFP) is a basic problem in network flow theory with several applications. The standard formulation of the MCFP assumes that input data are known precisely. In this paper we study a slight variation of this problem where the arc costs are imprecisely known. There are previous related results in the literature. In [3] the MCFP with stochastic arc costs is studied and solution methods are developed based on two optimality concepts: cycle marginal costs, and network equilibrium. In [1] the MCFP with interval arc costs

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is considered and two solution methods are introduced based on extensions of some efficient combinatorial algorithms for the MCFP. Also two performance indexes are used to measure the efficiency of these methods in simulations performed on different scenarios. In [2] the MCFP is established for fuzzy arc costs and, just as for the problem with interval arc costs, the proposed solution modifies the negative-cycle-canceling algorithm in order to allow the use of fuzzy numbers. In this work we solve both approaches of the MCFP: with interval and fuzzy arc costs. In both cases we solve the problem by transforming it into a conventional MCFP. We use the performance indexes introduced in [1] to compare both, the methodology in [1] and ours.

2 Intervals and fuzzy numbers

Intervals and fuzzy numbers are mathematical representations of imprecise quantities sucessfully applied to solve several problems in industrial engineering and operations research. Interval and fuzzy mathematics are generalizations of real aritmethic where numbers are replaced by intervals or fuzzy numbers. Some basic fuzzy numbers concepts are defined in Section 2.2. For all of the undefined concepts about fuzzy numbers we follow [5].

2.1 Intervals arithmetic

A closed interval in $\mathbb{R}$ is a set $[a_L, a_R] = \{x \in \mathbb{R} | a_L \leq x \leq a_R\}$ where a_L and a_R are the left and right *limits* of the interval. An interval $A = [a_L, a_R]$ is an *interval number* and is represented by $A^I = \langle a, a^w \rangle$ where $a = \frac{a_R + a_L}{2}$ and $a^w = \frac{a_R - a_L}{2} \geq 0$ are the *center* and the *width* of the interval number A^I respectively.

It is common to use the following two operations on intervals.

Definition 2.1.1. Let $\langle a, a^w \rangle$ and $\langle b, b^w \rangle$ be interval numbers and $\lambda \geq 0$ a real number. The addition of two interval numbers and the multiplication of an interval number by an escalar satisfies, respectively

$$\langle a, a^w \rangle + \langle b, b^w \rangle = \langle a + b, a^w + b^w \rangle \tag{1}$$

$$\lambda \langle a, a^w \rangle = \langle \lambda a, \lambda a^w \rangle = \langle a, a^w \rangle \lambda \tag{2}$$

A review of intervals and their algebraic properties appears in [4].

An ordering on a special kind of interval numbers was introduced by Hashemi et al. [1] based on a weighted scheme.

Definition 2.1.2. Let $\langle a, a^w \rangle$ and $\langle b, b^w \rangle$ be interval numbers and k, l real positive numbers. A relation $\leq_{k,l}$ on intervals is

$$\langle a, a^w \rangle \leq_{k,l} \langle b, b^w \rangle \Leftrightarrow ka + la^w \leq kb + lb^w.$$

The relation $\leq_{k,l}$ on interval numbers is reflexive, transitive and complete. The following definition and proposition establishes that this relation is an ordering on a particular subset of interval numbers for a special election of k and l.

Definition 2.1.3. Let $\mathcal{I}_{\mathbb{Q}} = \{\langle a, a^w \rangle | a, a^w \in \mathbb{Q}\}$ be the set of intervals with rational entries.

Proposition 2.1.4 ([1])**.** *Let π be a non-algebraic real positive number and $k = q_1\pi^{n_1}$, $l = q_2\pi^{n_2}$, where q_1, $q_2 \in \mathbb{Q}^+ - \{0\}$ are non-zero rational numbers and $n_1 \neq n_2 \in \mathbb{Z}^+$. Then, the relation $\leq_{k,l}$ provides a total order on $\mathcal{I}_{\mathbb{Q}}$.*

2.2 Fuzzy numbers arithmetic

A *fuzzy set* $\tilde{A}$ in the universe X is characterized by a *membership (characteristic) function* $\mu_{\tilde{A}} : X \to [0, 1]$ which associates with each point in X a "membership grade" in the interval $[0, 1]$. A *fuzzy number* $\tilde{A}$ is a fuzzy set in the universe $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$ where

1. $\mu_{\tilde{A}}$ is piecewise continuous
2. There exists a unique $x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x_0) = 1$
3. $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$ $\forall x_1, x_2 \in \mathbb{R}, \forall \lambda \in [0, 1]$

A fuzzy number $\tilde{M}$ is said to be an *LR fuzzy number* if and only if

$$\mu_{\tilde{M}}(x) = \begin{cases} L\Big(\dfrac{m-x}{m^L}\Big) & \text{for} \quad x \leq m, \quad m^L > 0 \\ R\Big(\dfrac{x-m}{m^R}\Big) & \text{for} \quad x \geq m, \quad m^R > 0 \end{cases}$$

where $L, R : \mathbb{R} \to [0, 1]$ are symmetric and non-increasing on $[0, +\infty)$ functions such that $L(0) = R(0) = 1$. Quantities m, m^L and m^R are called the mean value, and the left and right spreads of M respectively. Let us denote $\tilde{M} = (m, m^L, m^R)_{LR}$.

For LR fuzzy numbers there are two basic operations too [5].

Definition 2.2.1. Let $(a, a^L, a^R)_{LR}$ and $(b, b^L, b^R)_{LR}$ be LR fuzzy numbers and $\lambda \geq 0$ a real number. The addition of two LR fuzzy numbers and the multiplication of a fuzzy number by escalar satisfies, respectively

$$(a, a^L, a^R)_{LR} \;\tilde{+}\; (b, b^L, b^R)_{LR} = (a+b, a^L + b^L, a^R + b^R)_{LR} \tag{3}$$

$$\lambda(a, a^L, a^R)_{LR} = (\lambda a, \lambda a^L, \lambda a^R)_{LR} = (a, a^L, a^R)_{LR}\lambda \tag{4}$$

Analogous to interval numbers it is possible to define an ordering on a subset of LR fuzzy numbers.

Definition 2.2.2. Let $(a, a^L, a^R)_{LR}$ and $(b, b^L, b^R)_{LR}$ be LR fuzzy numbers and k, l, r real positive numbers. A relation $\leq_{k,l,r}$ on LR fuzzy numbers may be defined as

$$(a, a^L, a^R)_{LR} \leq_{k,l,r} (b, b^L, b^R)_{LR} \Leftrightarrow ka + la^L + ra^R \leq kb + lb^L + rb^R.$$

Definition 2.2.3. Let $\mathcal{LR}_{\mathbb{Q}} = \{(m, m^L, m^R)_{LR} | m, m^L, m^R \in \mathbb{Q}\}$ be the set of LR fuzzy numbers with rational entries.

Proposition 2.2.4 ([2])**.** *Let π be a non-algebraic real positive number and $k = q_1\pi^{n_1}$, $l = q_2\pi^{n_2}$ and $r = q_3\pi^{n_3}$ where q_1, q_2, $q_3 \in \mathbb{Q}^+ - \{0\}$ are non-zero rational numbers and $n_1 \neq n_2 \neq n_3 \in \mathbb{Z}^+$. Then, the relation $\leq_{k,l,r}$ provides a total order on $\mathcal{LR}_{\mathbb{Q}}$.*

3 The minimum imprecise-cost flow problem

Let $G = (N, A)$ be a directed graph where N and A are sets of nodes and arcs respectively. Each arc $(i, j) \in A$ has a cost $c_{i,j}$, and an integral capacity $u_{i,j}$. Each node $i \in N$ has a supply or demand represented as an integer b_i. If b_i is negative (resp. positive or zero) then the node i is a *demander* (resp. *supplier* or *transient*) node. Moreover, the sum of supplies and demands is assumed to be zero, i.e., $\sum_{i \in N} b_i = 0$.

The cost vector is denoted by $c = (c_{i,j})_{(i,j)\in A}$. Similarly $u = (u_{i,j})_{(i,j)\in A}$ and $b = (b_i)_{i\in N}$ denote respectively the capacities and supplies vector. The 5-tuple $\mathcal{N} = (N, A, u, c, b)$ is a *network*.

The minimum cost flow problem on network $\mathcal{N} = (N, A, u, c, b)$ consists in determining the flow $x_{i,j}$ on each arc $(i, j) \in A$ that solves the following problem.

Minimize:

$$\sum_{(i,j)\in A} c_{i,j}x_{i,j} \tag{5}$$

Subject to:

$$\sum_{\{j:(i,j)\in A\}} x_{i,j} - \sum_{\{j:(j,i)\in A\}} x_{j,i} = b_i \ \forall\ i \in N \tag{6}$$

$$0 \le x_{i,j} \le u_{i,j} \quad \forall\ (i, j) \in A \tag{7}$$

The flow vector $x = (x_{i,j})_{(i,j)\in A}$ is *feasible* if and only if it satisfies the contraints (6) - (7) and it is an *optimal* flow if its total transporting cost (5) is minimal among the costs of all feasible flows.

It is well-known that the MCFP can be solved efficiently in (strongly) polynomial time. The running times for several algorithmic implementations appears in [7, 8].

If instead of using numbers in the entries of the cost vector c we use interval numbers (resp. fuzzy numbers) a *minimum interval-cost flow problem* (resp. *minimum fuzzy-cost flow problem*), MICFP (resp. MFCFP) for short, is defined on network $\mathcal{N}$. We have a *minimum imprecise-cost flow problem* on network $\mathcal{N}$ if we have either a MICFP or a MFCFP on $\mathcal{N}$. The notion of feasible flow remains inaltered in this case, however the objective function

$$\sum_{(i,j)\in A} c_{i,j}x_{i,j}$$

is an interval or fuzzy number and the flow x is *optimal* if its cost is the minimum among all costs of feasible flows with respect to the $\le_{k,l}$ or $\le_{k,l,r}$ ordering.

4 The crisp model for a minimum imprecise-cost flow problem

Suppose we have a minimum imprecise-cost flow problem on a network $\mathcal{N} = (N, A, u, \hat{c}, b)$ with imprecise arc cost vector $\hat{c}$, and that all arc costs $\hat{c}_{i,j}$ are in $\mathcal{I}_{\mathbb{Q}}$ (resp. $\mathcal{LR}_{\mathbb{Q}}$). Let k and l (resp. k, l and r) be numbers such that the Hashemi's order is a total order on $\mathcal{I}_{\mathbb{Q}}$ (resp. $\mathcal{LR}_{\mathbb{Q}}$).

For each arc $(i, j) \in A$, let us define the *crisp cost* $\bar{c}_{i,j}$ as

$$\bar{c}_{i,j} = \begin{cases} kc_{i,j} + lc^w_{i,j}\ , & \text{if } \hat{c}_{i,j} = \langle c_{i,j}, c^w_{i,j} \rangle \\ kc_{i,j} + lc^L_{i,j} + rc^R_{i,j}\ , & \text{if } \hat{c}_{i,j} = (c_{i,j}, c^L_{i,j}, c^R_{i,j})_{LR} \end{cases}$$

so we can establish the *crisp model* associated to the original minimum imprecise-cost flow problem as

Minimize:

$$\sum_{(i,j)\in A} \bar{c}_{i,j} x_{i,j} \tag{8}$$

Subject to:

$$\sum_{\{j:(i,j)\in A\}} x_{i,j} - \sum_{\{j:(j,i)\in A\}} x_{j,i} = b_i \ \forall\ i \in N \tag{9}$$

$$0 \le x_{i,j} \le u_{i,j} \ \forall\ (i, j) \in A \tag{10}$$

which is a conventional minimum cost flow problem where the restrictions (9) - (10) are the same as in the imprecise problem but now with real arc costs in the objective (8) instead of imprecise values.

Proposition 4.1. *Let $x^* = (x^*_{i,j})_{(i,j)\in A}$ be a feasible flow which is an optimal solution for the crisp model (8) - (10) associated to a MFCFP. Then x^* also is an optimal solution for the MFCFP.*

Proof. Let k, l, and r be numbers inducing a total order on $\mathcal{LR}_{\mathbb{Q}}$ (as in Proposition 2.2.4).

Notice that a flow is a feasible flow in the crisp model if and only if it is a feasible flow in the MFCFP.

Let $x^* = (x^*_{i,j})_{(i,j)\in A}$ be an optimal flow for the crisp model. Then x^* is a feasible flow for the MFCFP too.

If the optimal solution for the MFCFP is $y^* = (y^*_{i,j})_{(i,j)\in A}$ and $\sum_{(i,j)\in A} \hat{c}_{i,j} y^*_{i,j} <_{k,l,r} \sum_{(i,j)\in A} \hat{c}_{i,j} x^*_{i,j}$. Then we have that

$$\sum_{(i,j)\in A} \left(c_{i,j}, c^L_{i,j}, c^R_{i,j}\right)_{LR} y^*_{i,j} <_{k,l,r} \sum_{(i,j)\in A} \left(c_{i,j}, c^L_{i,j}, c^R_{i,j}\right)_{LR} x^*_{i,j}$$

$$\Leftrightarrow \sum_{(i,j)\in A} \left(c_{i,j} y^*_{i,j}, c^L_{i,j} y^*_{i,j}, c^R_{i,j} y^*_{i,j}\right)_{LR} <_{k,l,r} \sum_{(i,j)\in A} \left(c_{i,j} x^*_{i,j}, c^L_{i,j} x^*_{i,j}, c^R_{i,j} x^*_{i,j}\right)_{LR}$$

$$\Leftrightarrow \left(\sum_{(i,j)\in A} c_{i,j} y^*_{i,j}, \sum_{(i,j)\in A} c^L_{i,j} y^*_{i,j}, \sum_{(i,j)\in A} c^R_{i,j} y^*_{i,j}\right)_{LR} <_{k,l,r}$$
$$\left(\sum_{(i,j)\in A} c_{i,j} x^*_{i,j}, \sum_{(i,j)\in A} c^L_{i,j} x^*_{i,j}, \sum_{(i,j)\in A} c^R_{i,j} x^*_{i,j}\right)_{LR}$$

$$\Leftrightarrow k \sum_{(i,j)\in A} (c_{i,j} y^*_{i,j}) \ + \ l \sum_{(i,j)\in A} (c^L_{i,j} y^*_{i,j}) \ + \ r \sum_{(i,j)\in A} (c^R_{i,j} y^*_{i,j}) <$$
$$k \sum_{(i,j)\in A} (c_{i,j} x^*_{i,j}) + l \sum_{(i,j)\in A} (c^L_{i,j} x^*_{i,j}) + r \sum_{(i,j)\in A} (c^R_{i,j} x^*_{i,j})$$

$$\Leftrightarrow \sum_{(i,j)\in A} \left(kc_{i,j} + lc^L_{i,j} + rc^R_{i,j}\right) y^*_{i,j} < \sum_{(i,j)\in A} \left(kc_{i,j} + lc^L_{i,j} + rc^R_{i,j}\right) x^*_{i,j}$$

$$\Leftrightarrow \sum_{(i,j)\in A} \bar{c}_{i,j} y^*_{i,j} < \sum_{(i,j)\in A} \bar{c}_{i,j} x^*_{i,j}$$

which is a contradiction to the optimality of x^* for the crisp model. Therefore, x^* is an optimal solution for the MFCFP. $\square$

Corollary 4.2. *Let $x^* = (x^*_{i,j})_{(i,j)\in A}$ be a feasible flow that is an optimal solution for the crisp model (8) - (10) associated to a MICFP. Then x^* also is an optimal solution for the MICFP.*

Proof. Analogous to proof of Proposition 4.1. $\square$

The last results are rewriten as follows.

Theorem 4.3. *The optimal solution for a minimum imprecise-cost flow problem is given by the optimal solution for the associated crisp model.*

Theorem 4.3 yields a simple method to solve a minimum imprecise-cost flow problem: we can directly use polynomial-time combinatorial algorithms to obtain the optimal solution for the associated crisp model. The optimal flow found is optimal for the original minimum imprecise-cost flow problem too.

5 Numerical simulation results

The crisp model methodology was tested on networks with interval arc costs consisting of 20 nodes and exactly 40 arcs. The nodes in these networks had two as average degree. These networks were randomly generated using the following procedure (see [3] for details).

1. Label the nodes in the network from 1 through n.
2. Set $b_1 = b$ and $b_n = -b$ for a positive integer b. The remaining nodes are transient nodes.
3. Generate $n-1$ directed arcs $(i, i+1)$ for all $i = 1, \cdots, n-1$ and set $u_{i,i+1} = b$, and $\hat{c}_{i,i+1} = \langle c, c^w \rangle$ where c and c^w are positive rational constants.
4. Generate the remainning $n+1$ arcs (i, j) by selecting their tail and head nodes randomly, each node should have the same probability to be selected, but parallel arcs and loops must be avoided. Arc's capacities, center costs, and cost widths are uniformly drawn from $[0, b]$, $[0, c]$ and $[0, c^w]$ respectively.

To measure the accuracy of the crisp model for the prediction of optimal flows in an imprecise environment we follow the scenario idea used in [1]. Let $\mathcal{N} = (N, A, u, \hat{c}, b)$ be a network with interval arc costs; the network $\mathcal{N}_s = (N, A, u, c^s, b)$ is a *scenario* of $\mathcal{N}$ if and only if each arc cost $c^s_{i,j}$ belongs to the interval arc cost $\langle c_{i,j}, c^w_{i,j} \rangle$ for all $(i, j) \in A$. Two performance indexes for a set of scenarios S of $\mathcal{N}$ are defined in [1] as follows.

Let $V^*(\mathcal{N}')$ be the optimal value of a instance of one of the problems MCFP, MICFP or MFCFP on a given network $\mathcal{N}'$. An instance of the MICFP on network $\mathcal{N}$ solved by the crisp model has an interval as optimal cost, i.e., $V^*(\mathcal{N}) = \langle V, V^w \rangle$ for an interval $\langle V, V^w \rangle$. The *first performance index* denoted $I_1(\mathcal{N})$, is the proportion of scenarios of $\mathcal{N}$

	$\|S\|$	1000		3000		5000	
k	l	I_1	I_2	I_1	I_2	I_1	I_2
1	4π	1.0	0.0	1.0	0.0	1.0	6.0
1	3π	1.0	1.0	1.0	1.0	1.0	0.0
1	2π	1.0	5.0	1.0	5.0	1.0	0.0
1	π	1.0	0.0	1.0	0.0	1.0	0.0
1	$\pi/2$	1.0	4.0	1.0	3.9	1.0	7.0
1	$\pi/3$	1.0	1.0	1.0	1.0	1.0	9.0
1	$\pi/4$	1.0	0.0	1.0	8.9	1.0	18.0

Table 1: Results of the MICFP in the random networks with 20 nodes.

whose optimal costs are in the interval $V^*(\mathcal{N})$. More precisely,

$$I_1(\mathcal{N}) = \frac{|\{\mathcal{N}_s \in S : V^*(\mathcal{N}_s) \in V^*(\mathcal{N})\}|}{|S|}.$$

As much as this index is close to one, the crisp model predicts the cost of shipment more accurately.

Let $x = (x_{i,j})_{(i,j)\in A}$ be the optimal flow in the MICFP on network $\mathcal{N}$ obtained by solving its associated crisp model, and $x^s = (x^s_{i,j})_{(i,j)\in A}$ be the optimal flow for scenario $\mathcal{N}_s$. The *second performance index* denoted $I_2(\mathcal{N})$, is the maximum difference between arc flow entries in x and in x^s normalized per cost unit. More precisely,

$$I_2(\mathcal{N}) = \max_{\mathcal{N}_s \in S} \max_{(i,j)\in A} \left\{ |x_{i,j} - x^s_{i,j}| \frac{c^s_{i,j}}{\max\limits_{(i,j)\in A} c^s_{i,j}} \right\}.$$

The method yields a better solution as the second index is close to zero.

In Table 1 we report the results produced by the crisp model methodology to solve some instances of the MICFP's and the values obtained for $I_1(\mathcal{N})$ and $I_2(\mathcal{N})$. We choose similar values for k and l as used in [1, 2], and for each pair of values k and l a random network with interval arc costs is created and then a set S of scenarios is generated. Finally the indexes I_1 and I_2 are calculated.

Now let us consider the MCFP on network $\mathcal{N}$ where the arc costs are all LR fuzzy numbers, i.e., a MFCFP. For every $\alpha \in [0, 1]$ the *α-level set* (or *α-cut*) of a fuzzy set $\tilde{A}$ is the ordinary set $\tilde{A}_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ and when $\tilde{A}$ is an LR fuzzy number $\tilde{A}_\alpha$ is always a closed interval.

		α	0.0		0.25		0.5		0.75	
k	l	r	I_1	I_2	I_1	I_2	I_1	I_2	I_1	I_2
1	4π	$16\pi^2$	1.0	9.5	1.0	0.0	1.0	12.0	1.0	0.0
1	3π	$9\pi^2$	1.0	11.2	1.0	0.0	1.0	0.0	1.0	0.0
1	2π	$4\pi^2$	1.0	10.0	1.0	3.0	1.0	0.0	1.0	0.0
1	π	π^2	1.0	0.0	1.0	0.0	1.0	0.0	1.0	1.0
1	$\pi/2$	$\pi^2/4$	1.0	12.0	1.0	0.0	1.0	4.0	1.0	0.0
1	$\pi/3$	$\pi^2/9$	1.0	20.0	1.0	0.0	1.0	0.0	1.0	0.0
1	$\pi/4$	$\pi^2/16$	1.0	9.0	1.0	4.0	1.0	15.0	1.0	0.0

Table 2: Results for 5000 α-escenarios in the MFCFP on random networks with 20 nodes.

Extending the scenario idea for networks with fuzzy arc costs we can take each cost $c^s_{i,j}$ of scenario $\mathcal{N}_s$ in the interval defined by an α-cut of the fuzzy cost $\tilde{c}_{i,j}$. Actually this gives us an *α-scenario* $\mathcal{N}_s(\alpha)$ for each possible value of α.

By a similar procedure applied on a randomly generated network $\mathcal{N}$ with triangular fuzzy arc costs we measured the performance indexes I_1 and I_2 for several α-scenarios of the MFCFP. The results obtained for some possibility level α are show in Table 2.

In both, Tables 1 and 2, we obtained $I_1 = 1$ for all experiments. This improves the results reported in [1] where a value of one was never reached for I_1. On the other hand, in the same reference no value reported for I_2 is zero, but there are several entries in Tables 1 and 2 where this optimal value is reached. Thus, in more than 40% of the entries in Table 1 we obtained the best possible value for I_2 by using the crisp model and the same happened for more than 50% of entries in Table 2. The authors of [1] never got zero values. The combination $I_1 = 1$ and $I_2 = 0$ appears in 48% of the results reported in Tables 1 and 2, hence in almost 50% of all experiments performed we obtained accurate solutions.

6 Conclusion

Minimum cost flow problems are important in network optimization due to their wide range of applications. In this paper we assumed that arc costs in the network are imprecise values that could be described by intervals as well as by LR fuzzy numbers. We solved the minimum cost flow problem with imprecise arc costs by applying the crisp model methodology and transforming it into a crisp minimum cost flow problem. This transformation was based on the total order introduced in [1, 2] for a special kind of intervals and fuzzy numbers. Although in this paper we choose the k, l and r values accordingly to the recommendations in [1, 2], it is an important question to ask for a proper way to do this selection, because in a floating arithmetic system it is impossible to represent non-algebraic numbers even if they are computable.

Our choosing of k, l and r as powers (including exponent 0) of rational multiples of π is motivated (and supported) by the positive results reported in Tables 1 and 2. Yet, we believe a deeper study is in order because the use of a floating point system has radical consequences on the truthfulness of Propositions 2.1.4 and 2.2.4.

Finally, numerical simulation showed that the use of the crisp model methodology improves upon the extension of the combinatorial algorithm proposed in [1] and [2], and it is at least comparable to the existing methods.

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