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An upper bound on the size of irreducible quadrangulations *

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Abstract

Let S be a closed surface with Euler genus $\gamma(S)$. A quadrangulation G of a closed surface S is irreducible if it does not have any contractible face. Nakamoto and Ota gave a linear upper bound for the number n of vertices of G in terms of $\gamma(S)$. By extending Nakamoto and Ota's method we improve their bound to $n \leq 159.5\gamma(S) - 46$ for any closed surface S.

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1 Introduction

The *orientable closed surface* M_g with *genus* g is the sphere with g handles attached. The *non-orientable closed surface* N_g with *genus* g is the sphere with g cross-caps attached. The *Euler genus* of these surfaces is $\gamma(M_g) = 2g$ for the orientable surface M_g and $\gamma(N_g) = g$ for the non-orientable surface N_g.

Let G be a *simple graph*, that is, a graph without loops or parallel edges. The *orientable genus* $\bar{\gamma}(G)$ of G is defined as the least g such

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that G is embeddable in M_g and the *non-orientable genus* $\tilde{\gamma}(G)$ of G is defined as the least g such that G is embeddable in N_g. The *Euler genus* $\gamma(G)$ of G is defined to be $\gamma(G) = \min\{2\bar{\gamma}(G), \tilde{\gamma}(G)\}$. Note that $\gamma(G) = \min\{\gamma(S) : G \text{ is embeddable in } S\}$.

A *quadrangulation* G of a closed surface S is a graph 2-cell embedded on S in such a way that all faces of G are quadrangles.

Let G be a quadrangulation of a closed surface S and let $abcd$ be a face of G. We say that $abcd$ is *contractible* if we can obtain a new quadrangulation by identifying a with c and deleting edges ab and cd, see Figure 1. A quadrangulation G is said to be *irreducible* if G has no contractible face. The size of an irreducible quadrangulation can be measured in terms of its number of vertices, edges, or faces. By Euler's formula these are all equivalent and we have chosen to measure the size in terms of the number n of vertices.

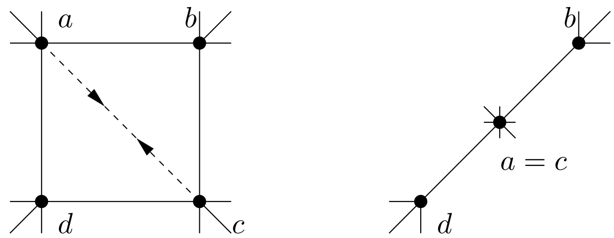

Figure 1: Contracting face *abcd*.

Nakamoto and Ota [3] gave a linear upper bound for the number n of vertices of an irreducible quadrangulation G in terms of $\gamma(S)$, namely $n \leq 186\gamma(S) - 64$.

In this paper we prove an upper bound of $n \leq 159.5\gamma(S) - 46$ for the size of our quadrangulations by extending Nakamoto and Ota's method.

2 Preliminaries

We use the following bound on the Euler genus of a 1− or 2−sum of graphs.

Lemma 2.1 (Miller [1]). *Let G_1 and G_2 be two graphs and let $G := G_1 \cup G_2$. If G_1 and G_2 have at most two common vertices, then $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$.*

From now on, let S be either M_g or N_g with $g \geq 1$ and let G be an irreducible quadrangulation of S. Let G' be the graph embedded on S obtained from G by adding a vertex of degree four into each of its faces and joining it to the vertices of the corresponding face, see Figure 2.

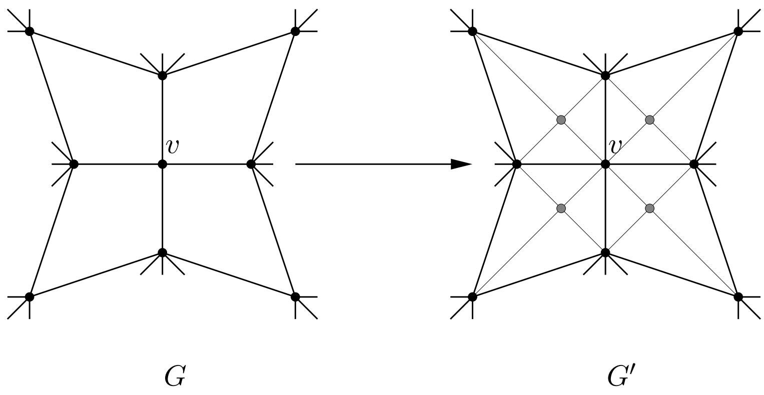

Figure 2: Construction of G'.

For $v \in V(G)$ let H_v be the subgraph of G' induced by v, the vertices of the incident faces to v in G and the vertices added to the incident faces to v. Let $N_G(v)$ be the set of adjacent vertices to v in G.

Nakamoto and Ota [2] proved the following result for $\deg_G(v) \leq 4$.

Lemma 2.2. *Let G be an irreducible quadrangulation of a closed surface S and let v be a vertex of G. Then $\gamma(H_v) \geq 1$.*

Proof. Since G is irreducible and S is not the sphere, it follows that G has no vertex of degree less than three. Let v be a vertex of G of degree $d \geq 3$ and let $W_v := v_0e_0v_1e_1 \dots v_{2d-1}e_{2d-1}v_{2d}$ be a closed walk in G such that $v_0, v_2, \dots, v_{2d-2}$ are the neighbors of v in clockwise direction and $v\,v_{2i}v_{2i+1}v_{2i+2}$ is a face of G for $i = 0, 1, \dots, d-1$. Since G is irreducible every v_{2i+1} must be adjacent or equal to some v_{2j} with $j \neq i, i+1$. We denote by w_{2i+1} the new vertex of degree four added to the face $v\,v_{2i}v_{2i+1}v_{2i+2}$ in G. Let $v_m, v_n \in W_d$, we define $\operatorname{dist}(v_m, v_n) := \min\{|m-n|-1, 2d-|m-n|-1\}$, for $m \neq n$.

Let $v_\alpha, v_\beta \in W_d$ be vertices such that $v_\alpha \in N_G(v)$, $v\,v_{\beta-1}v_\beta v_{\beta+1} \in F(G)$, v_α is adjacent or equal to v_β and $\operatorname{dist}(v_\alpha, v_\beta) > 0$ is minimal. Without loss of generality we can assume that $\beta = 1$, $\alpha = 2k+2$, and $\operatorname{dist}(v_\alpha, v_\beta) = 2k$.

Since G is irreducible v_{2k+1} is adjacent or equal to some neighbor of v, namely v_{2j}, with $k+1 < j < d$ because $\mathrm{dist}(v_\alpha, v_\beta) = 2k$ is minimal, see Figure 3.

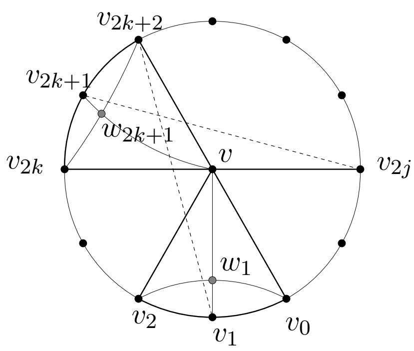

Figure 3: Vertex of degree $d \geq 3$.

Therefore we have a subdivision of $K_{3,3}$ with partition

$$\{v, w_{2k+1}, w_1\} \cup \{v_{2k+2}, v_{2k}, v_{2j}\}.$$

See Figure 4. □

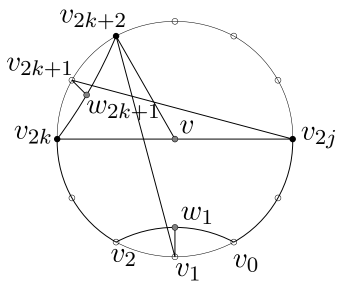

Figure 4: Subdivision of $K_{3,3}$.

We say that a set I of vertices of G is *face-independent* if no two vertices in I are incident to the same face of G. Nakamoto and Ota proved the following result for $k = 4$ [2].

Lemma 2.3. *Let G be an irreducible quadrangulation of a closed surface S and let $k \geq 3$ be an integer. For each $i \geq 3$ let V_i be the set of vertices* *of degree i of G. Then there exists an independent set $X \subseteq V_3 \cup \cdots \cup V_k$ such that*

$$|X| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1}.$$

Proof. Let X_3 be a maximal face-independent subset of V_3. For $i = 4, \ldots, k$ let X_i be a maximal face-independent subset from

$$V_i - \bigcup_{j=3}^{i-1} A_{i,j}$$

where $A_{i,j}$ is the set of vertices of degree i that are incident to a face which is incident to a vertex in X_j. We claim that $X = \bigcup_{i=3}^{k} X_i$ satisfies the required property. Counting the vertices in the incident faces of each vertex x in X_i (including vertex x) we obtain

$$(2i+1)|X_i| \quad \geq \quad |V_i| + \sum_{j=i+1}^{k} |A_{j,i}| - \sum_{j=3}^{i-1} |A_{i,j}| \qquad \text{for every } 3 \leq i \leq k.$$

Therefore

$$|X| = \sum_{i=3}^{k} |X_i| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1} + \sum_{i=3}^{k-1} \sum_{j=i+1}^{k} \frac{|A_{j,i}|}{2i+1} - \sum_{i=4}^{k} \sum_{j=3}^{i-1} \frac{|A_{i,j}|}{2i+1}.$$

Observe that $A_{i,j} = \emptyset$ for $i \leq j$. Therefore

$$\sum_{i=3}^{k-1} \sum_{j=i+1}^{k} \frac{|A_{j,i}|}{2i+1} - \sum_{i=4}^{k} \sum_{j=3}^{i-1} \frac{|A_{i,j}|}{2i+1} = \sum_{i=3}^{k-1} \sum_{j=i+1}^{k} \left[\frac{|A_{j,i}|}{2i+1} - \frac{|A_{j,i}|}{2j+1} \right] \geq 0$$

since $j > i$. We conclude that

$$|X| \geq \sum_{i=3}^{k} \frac{|V_i|}{2i+1}. \qquad \square$$

3 Main Result

Theorem 3.1. *Let G be an irreducible quadrangulation of a closed surface S with n vertices. Then*

$$n \leq 159.5\gamma(S) - 46.$$

Proof. Let m and f be the number of edges and faces of G, respectively. By Euler's formula

$$n - m + f = 2 - \gamma(S).$$

Since G is a quadrangulation we have that $4f = 2m$ and therefore

$$4n - 2m = 8 - 4\gamma(S).$$

Since $\sum_{i\geq 3} |V_i| = n$ and $\sum_{i\geq 3} i|V_i| = 2m$ we have that

$$3n + \sum_{i\geq 3}(1-i)|V_i| = 8 - 4\gamma(S).$$

Let $k \geq 4$ be an integer (to be chosen later). By adding and substracting $kn = k\sum_{i\geq 3} |V_i|$ we obtain

$$(3-k)n + \sum_{i\geq 3}(k+1-i)|V_i| = 8 - 4\gamma(S).$$

Thus

$$\sum_{i=3}^{k}(k+1-i)|V_i| \geq (k-3)n - 4\gamma(S) + 8. \tag{1}$$

Let X be an independent set as in Lemma 2.3 and define

$$Y := \{y \in V(G) - X | y \in N_G(x) \text{ for some } x \in X\}.$$

Consider the bipartite graph B with bipartition X and Y, where $xy \in E(B)$ for $x \in X$, $y \in Y$ if and only if $xy \in E(G)$.

Let $X' := \{v_1, v_2, \ldots, v_r\}$ be a maximal subset of X satisfying the following condition:

$$\left| \left\{ \bigcup_{1\leq i<j} N_B(v_i) \right\} \cap N_B(v_j) \right| \leq 2, \text{ for each } j = 1, 2, \ldots, r.$$

In other words, for each $2 \leq j \leq r$, v_j has at most two common neighbors with $v_1, v_2, \ldots, v_{j-1}$. See Figure 5.

By Lemma 2.1 and Lemma 2.2 we obtain

$$\gamma\left(\bigcup_{i=1}^{r} H_{v_i}\right) \geq \sum_{i=1}^{r} \gamma(H_{v_i}) \geq r = |X'|.$$

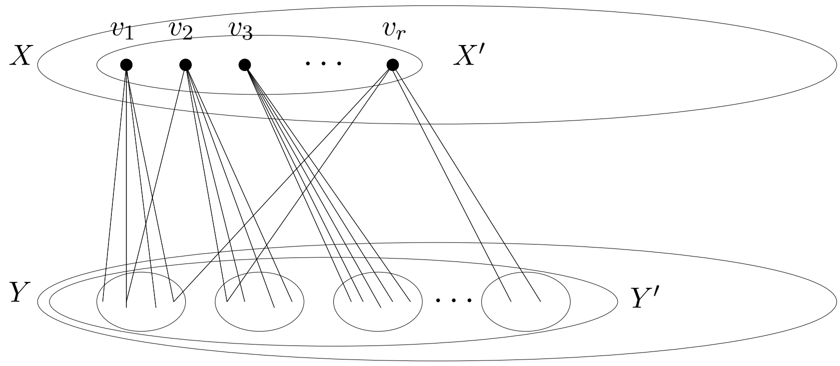

Figure 5: The sets X, X', Y, and Y'.

Since $\bigcup_{i=1}^{r} H_{v_i}$ is a subgraph of G', it is embeddable in S, thus

$$\gamma(S) \geq \gamma\left(\bigcup_{i=1}^{r} H_{v_i}\right),$$

and it follows that

$$|X'| \leq \gamma(S). \tag{2}$$

Now define $Y' := \{y \in Y | y \in N_B(v) \text{ for some } v \in X'\}$. Let M be the subgraph of B induced by $X \cup Y'$. Since M is a subgraph of G it is embeddable in S, therefore

$$|V(M)| - |E(M)| + |F(M)| \geq 2 - \gamma(S).$$

Since M is bipartite each of its faces has at least 4 edges, therefore $4|F(M)| \leq 2|E(M)|$. Hence we have

$$2|V(M)| - |E(M)| \geq 4 - 2\gamma(S). \tag{3}$$

By maximality of X', each vertex $v \in X - X'$ has at least three neighbors in Y'. There are at least $|Y'|$ edges between X' and Y'. Hence

$$|E(M)| \geq 3(|X| - |X'|) + |Y'|.$$

By replacing $|V(M)| = |X| + |Y'|$ and $|E(M)|$ in inequality (3), we obtain

$$\begin{aligned}
4-2\gamma(S) &\le 2|X|+2|Y'|-3(|X|-|X'|)-|Y'| \\
&\le -|X|+|Y'|+3|X'| \\
&\le -|X|+(2k+3)|X'| \qquad \text{since } |Y'|\le 2k|X'| \\
&\le -\sum_{i=3}^{k}\frac{|V_i|}{2i+1}+(2k+3)|X'| \qquad \text{by Lemma 2.3.}
\end{aligned}$$

Let n_k be the smallest integer such that

$$\frac{n_k}{2i+1}\ge k+1-i, \text{ for every } 3\le i\le k.$$

Since $(2i+1)(k-i+1)$ has a unique maximum, we take this value as n_k, namely

$$n_k := \begin{cases} 14 & \text{if } k=4 \\ \dfrac{(k+1)(k+2)}{2} & \text{if } k\ge 5. \end{cases}$$

Thus we obtain

$$\begin{aligned}
4-2\gamma(S) &\le -\frac{1}{n_k}\sum_{i=3}^{k}\frac{n_k}{2i+1}|V_i|+(2k+3)|X'| \\
&\le -\frac{1}{n_k}[(k-3)n+8-4\gamma(S)]+(2k+3)|X'|
\end{aligned}$$

and therefore

$$\frac{(k-3)n+8+4n_k-(2k+3)n_k|X'|}{4+2n_k}\le\gamma(S). \tag{4}$$

Thus, (2) provides a good bound for $\gamma(S)$ when $|X'|$ is *large* and (4) provides a good bound when $|X'|$ is *small*. These two bounds are the same when their left-hand sides are equal, that is, when

$$|X'|=\frac{(k-3)n+4n_k+8}{(2k+5)n_k+4}.$$

In particular, from (2) we obtain

$$\frac{(k-3)n+4n_k+8}{(2k+5)n_k+4}\le\gamma(S),$$

that is

$$n \leq f(k)\gamma(S) - g(k),$$

where $f(k) = \dfrac{(2k+5)n_k + 4}{k-3}$ and $g(k) = \dfrac{4n_k + 8}{k-3}$. A straightforward calculation shows that $f(k)$ attains its minimum at $k = 5$, therefore

$$n \leq 159.5\gamma(S) - 46. \qquad \square$$

Observe that for $k = 4$ we obtain $n \leq 186\gamma(S) - 64$, this is the bound obtained by Nakamoto and Ota [3].

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