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Real options on consumption in a small open monetary economy: a stochastic optimal control approach

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Abstract

This paper is aimed to develop a stochastic model of a small open monetary economy where risk-averse agents have expectations of the exchange-rate dynamics driven by a mixed diffusion-jump process. The size of a possible exchange-rate depreciation is supposed to have an extreme value distribution of the Fréchet type. Under this framework, an analytical solution of the price of the real option of waiting when consumption can be delayed (a claim that is not traded) is derived. Finally, a Monte Carlo simulation experiment is carried out to obtain numerical approximations of the real option price.

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1 Introduction

Real options have been attracting an increasing attention in economic theory and mathematical economy; see, for instance: Beck and Stockman (2005) studying money as a real option, Strobel (2005) examining monetary integration and inflation preferences through real options, Henderson and Hobson (2002) analyzing real options with constant relative risk aversion, and Foote and Folta (2002) dealing with temporary labor as a real option, among others. The main issue associated with real options is how to value a non-traded contingent claim[1].

[1] We refer the reader to the two classical books in real options: Dixit and Pindyck (1994), and Schwartz and Trigeorgis (2001).

In this paper, the underlying asset of the option is the price of money in terms of goods, that is, the consumer's adquisitive power, which is a non-traded asset (there is no markets for trading purchasing power) and, therefore, the option becomes also a non-traded claim. The real option we will be valuing is the option to set off consumption when the adquisitive power reaches a certain threshold in a future given date; otherwise the individual will have to wait. Even though there is no market for this contingent claims, its value provides an idea of how much the individual is willing to pay for activating consumption. It should be clear, for the reader, that this approximation for valuing derivatives is different from the complete market approach, developed in the Black-Scholes-Merton theory, in which the contingent claim can be replicated by a portfolio that combines stock (available in a stock market) and bonds (available in a credit market).

In this research, by generalizing Henderson and Hobson's (2002) paper, we value the real option of waiting when consumption can be delayed in a small open monetary economy with a representative, competitive, and risk-averse consumer. To reach this goal, Merton's model (1976) is extended by including an extreme value distribution for the jump size of the underlying; an analytical solution for the price of the derivative is obtained. It should be also emphasized that the proposed valuing procedure differs from that in Venegas-Martínez (2005), which is based on Bayesian inference, by now using the von Neumann-Morgenstern expected utility framework, which, of course, provides further economic and financial intuition.

This paper develops a stochastic economy that explicitly recognizes the role of extreme or exceptional movements in the dynamics of the nominal exchange rate. It is assumed that the exchange-rate dynamics follows a mixed diffusion-jump process where the size of an upward jump is supposed to have an extreme value distribution of the Fréchet type. In this case, the underlying non-traded asset is the price of money in terms of goods. Using this stochastic setting and assuming identical rational consumers with logarithmic preferences (risk-averse individuals), the price of such a real option is characterized as the solution of a (partial) differential-integral equation with boundary conditions. In fact, we provide an analytical solution of the value of such a real option. Finally, several Monte Carlo simulation experiments are carried out to get numerical approximations of the real option price.

Even though this work was, mainly, intended for mathematicians (dealing with mathematical finance) and economists (dealing with mathematical economy), it was not written in terms of definitions, theorems, propositions, and remarks. I should say that when I started writing this paper I did it in a prose style (there was not a specific reason then), and when I finished I realized that the result was a good story. I apologize and hope my colleagues enjoy this story as much as I did.

The paper is organized as follows. In the next section, we work out a one-good, cash-in-advance, stochastic economy where agents have expectations of the exchange-rate dynamics driven by a mixed diffusion-jump process and the size of a possible exchange-rate depreciation is supposed to have an extreme value distribution. Through section 3, we undertake the consumer's decision problem. In section 4, we deal with valuing the real option of delaying consumption. In section 5, we provide numerical approximations of the real option price. Finally, in section 6, we present conclusions, acknowledge limitations, and make suggestions for further research.

2 Structure of the model

Let us consider a small open monetary economy populated by infinitely lived identical households in a world with a single consumption good internationally tradable. The main assumptions on the economy resemble those from Venegas-Martínez (2001), (2006a) and (2006b), and they will be described in what follows.

2.1 Purchasing power parity and exchange rate dynamics

We assume that the consumption good is freely traded, and its domestic price level, P_t, is determined by the purchasing power parity condition, namely, we assume that the good in the economy is freely traded and its domestic price level, P_t, is determined by the purchasing power parity condition, namely,

$$P_t = P_t^* e_t, \tag{1}$$

where P_t^* is the foreign-currency price of the good in the rest of world, and e_t is the nominal exchange rate. Throughout the paper, we will assume, for convenience, that P_t^* is equal to 1. We also suppose that the initial value of the exchange-rate, e_0, is known and equal to 1.

In what follows, we will suppose that the ongoing uncertainty in the dynamics of the expected exchange rate, and therefore in the inflation rate, is generated by a geometric Brownian motion combined with a

Poisson process where the size of a forward jump is driven by extreme value distributions of the Fréchet type, that is,

$$\frac{\mathrm{d}e_t}{e_t} = \frac{\mathrm{d}P_t}{P_t} = \mu \mathrm{d}t + \sigma \mathrm{d}W_t + Z\mathrm{d}N_t \tag{2}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $(W_t)_{t \geq 0}$ is a Brownian motion defined on a fixed probability space $(\Omega, \mathcal{F}, \mathrm{P}_W)$, and $\mathrm{d}N_t$ is a Poisson process with intensity parameter λ. From now on, it will be supposed that $\mathrm{Cov}(\mathrm{d}W_t, \mathrm{d}N_t) = 0$. Even though it is easy to incorporate downward jumps by adding a second Poisson process in (2) multiplied by a Weibull distribution, for the sake of simplicity we will keep the analysis only for upward jumps (*cf.* Venegas-Martínez (2006c)). Moreover, extreme downward movements in the exchange rate or in inflation have never been observed, this situation would be quite atypical. The size of an upward jumps is defined by

$$Z = \frac{1}{1 - X^{-\alpha}} - 1, \quad X > 0, \ \alpha > 0,$$
$$X = \frac{Y - \nu}{\kappa}, \quad \kappa, \ \nu > 0,$$

where Y is a Fréchet random variable with parameters α, ν and $\kappa > 0$. Clearly, the quantity Z remains positive. The cumulative distribution function of Y is given by:

$$F_Y(y) = \begin{cases} 0, & y < \nu, \\ \exp\left\{ -\left(\frac{y-\nu}{\kappa}\right)^{-\alpha} \right\}, & y \geq \nu. \end{cases} \tag{3}$$

The corresponding density of Y satisfies:

$$f_Y(y) = \frac{\alpha}{\kappa} F_Y(y) \exp\left(\frac{y-\nu}{\kappa}\right)^{-\alpha} \tag{4}$$

On the other hand, since the number of expected upward jumps in the exchange rate, per unit of time, follows a Poisson process $\mathrm{d}N_t$ with intensity λ, we have that

$$\mathrm{P}_N\{\text{one unit jump during } \mathrm{d}t\} = \mathrm{P}_N\{\mathrm{d}N_t = 1\} = \lambda \mathrm{d}t$$

and

$$\mathrm{P}_N\{\text{more than one unit jump during } \mathrm{d}t\} = \mathrm{P}_N\{\mathrm{d}N_t > 1\} = o(\mathrm{d}t),$$

so that

$$\mathrm{P}_N\{\text{no jump during } \mathrm{d}t\} = 1 - \lambda \mathrm{d}t + o(\mathrm{d}t),$$

where $o(\mathrm{d}t)/\mathrm{d}t \to 0$ as $\mathrm{d}t \to 0$.

2.2 A cash-in-advance constraint

Consider a cash-in-advance constraint of the Clower type:

$$\psi m_t = c_t, \tag{5}$$

where m_t is the demand for real cash balances, c_t is the demand for consumption, and $\psi^{-1} > 0$ is the time that money must be held to in order to finance consumption. The constant ψ applies uniformly at all time t. Condition (5) is critical in linking the exchange-rate dynamics with consumption. An economic interpretation of a cash-in-advance constraint is that money is needed to buy consumption goods. Notice that when $\psi = 1$ the agent is forced to maintain his demand for money balances in the same proportion of demanded goods. Moreover, if we state the following link between m_t and c_s:

$$m_t = \int_t^{t+\psi^{-1}} c_s \mathrm{d}s,$$

where $\psi^{-1} > 0$ is the time that money must be held to buy consumption goods, then

$$m_t = \int_t^{t+\psi^{-1}} c_s \mathrm{d}s = \frac{c_t}{\psi} + o\left(\frac{1}{\psi}\right).$$

If the error term $o(1/\psi)$ is neglected, it follows that $m_t\psi = c_t$, as in (5).

2.3 The return rates of non-traded and traded assets

Let $S_t = 1/P_t$ the price of money in terms of goods, a non-traded asset, and $V = V(S_t, t)$ the price of a European call option on S_t; a non-traded contingent claim. Suppose also that there is a real bond of price b_t that pays a constant real interest rate r (*i.e.*, it pays r units of the consumption good per unit of time). Thus, the consumer's real wealth, x_t, is given by

$$x_t = S_t + V(S_t, t) + b_t, \tag{6}$$

where x_0 is exogenously determined. The stochastic rate of return of S_t, $\mathrm{d}R_S$, is obtained by applying Itô's lemma to the inverse of the price level, with (2) as the underlying process, that is,

$$\begin{aligned}\mathrm{d}\left(\frac{1}{P_t}\right) =& \left[-\left(\frac{1}{P_t^2}\right)\mu P_t + \tfrac{1}{2}\left(\frac{2}{P_t^3}\right)\sigma^2 P_t^2\right]\mathrm{d}t - \left(\frac{1}{P_t^2}\right)\sigma P_t \mathrm{d}W_t \\ &+ \left(\frac{-X^{-\alpha}+1}{P_t} - \frac{1}{P_t}\right)\mathrm{d}N_t \\ =&\frac{1}{P_t}\left[\left(-\mu+\sigma^2\right)\mathrm{d}t - \sigma \mathrm{d}W_t - X^{-\alpha}\mathrm{d}N_t\right].\end{aligned} \tag{7}$$

Hence, the stochastic rate of return of S_t is given by

$$\mathrm{d}R_S = \left(\sigma^2 - \mu\right)\mathrm{d}t - \sigma \mathrm{d}W_t - X^{-\alpha}\mathrm{d}N_t. \tag{8}$$

Observe now that the stochastic rate of return of S_t, $\mathrm{d}R_S = \mathrm{d}S_t/S_t$, can be rewritten as[2]

$$\mathrm{d}R_S = \phi \mathrm{d}t + \sigma \mathrm{d}W_t + \xi \mathrm{d}N_t, \tag{9}$$

where $\phi = \sigma^2 - \mu$ and $\xi = -X^{-\alpha}$. If $V = V(S_t, t)$ denotes the value of the option, then Itô's lemma leads to

$$\begin{aligned}\mathrm{d}V =& \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t}\phi S_t + \tfrac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\sigma^2 S_t^2\right)\mathrm{d}t \\ &+ \frac{\partial V}{\partial S_t}\sigma S_t \mathrm{d}W_t + \left[V(S_t(\xi+1), t) - V(S_t, t)\right]\mathrm{d}N_t\end{aligned}$$

or,

$$\mathrm{d}V = \phi_V V \mathrm{d}t + \sigma_V V \mathrm{d}W_t + \xi_V V \mathrm{d}N_t, \tag{10}$$

where

$$\phi_V = \frac{1}{V}\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t}\phi S_t + \tfrac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\sigma^2 S_t^2\right),$$

$$\sigma_V = \frac{1}{V}\frac{\partial V}{\partial S_t}\sigma S_t$$

and

$$\xi_V = \frac{1}{V}\left[V(S_t(\xi+1), t) - V(S_t, t)\right].$$

[2]Another approach for the dynamics of the underlying asset can be found in Venegas-Martínez (2005).

3 The household's decision problem

The consumer's real wealth stochastic accumulation in terms of the portfolio shares, $w_{1t} = S_t/x_t$, $w_{2t} = V/x_t$, $1 - w_{1t} - w_{2t} = b_t/x_t$, and consumption, c_t, is given by

$$\mathrm{d}x_t = x_t w_{1t} \mathrm{d}R_S + x_t w_{2t} \mathrm{d}R_V + x_t(1 - w_{1t} - w_{2t}) r \mathrm{d}t - c_t \mathrm{d}t,$$

with x_0 exogenously determined. In this equation, $\mathrm{d}R_V \equiv \mathrm{d}V/V$. Thus, by substituting (9) and (10) in the above expression, the budget constraint can be rewritten as

$$\begin{aligned} \mathrm{d}x_t = x_t \Big[& (r + (\gamma - r) w_{1t} + (\phi_V - r) w_{2t}) \mathrm{d}t + (w_{1t}\sigma + w_{2t}\sigma_V) \mathrm{d}W_t \\ & + (w_{1t}\xi + w_{2t}\xi_V) \mathrm{d}N_t \Big], \end{aligned} \tag{11}$$

where $\gamma = \sigma^2 - \mu - \psi = \phi - \psi$.

3.1 The utility index

The von Neumann-Morgenstern utility at time $t = 0$, v_0, of the competitive risk-averse consumer is assumed to have the time-separable form:

$$v_0 = \mathrm{E}_0 \left[\int_0^\infty \log(c_t)\ e^{-rt} \mathrm{d}t \right], \tag{12}$$

where E_0 is the conditional expectation on all available information at $t = 0$. To avoid unnecessary complex dynamics in consumption, we assume that the agent's subjective discount rate is consistent with the constant real international rate of interest, r. We consider the logarithmic utility function in order to derive closed-form solutions and make the subsequent analysis more tractable.

3.2 The first order conditions

The Hamilton-Jacobi-Bellman equation for the stochastic optimal control problem of maximizing utility, with $\log(c_t) = \log(\psi x_t w_{1t})$ and subject to (11), is given by

$$\max_{w_{1t},w_{2t}} H(w_t;x_t,t) \equiv \max_{w_{1t},w_{2t}} \Bigg\{ \log(\psi x_t w_{1t})e^{-rt} \\ + I_x(x_t,t)x_t[r+(\gamma-r)w_{1t}+(\phi_V-r)w_{2t}] \\ + I_t(x_t,t)+\tfrac{1}{2}I_{xx}(x_t,t)x_t^2(w_{1t}\sigma+w_{2t}\sigma_V)^2 \\ + \lambda \mathrm{E}_\xi\Big[I\Big(x_t(w_{1t}\xi+w_{2t}\xi_V+1),t\Big)-I(x_t,t)\Big]\Bigg\}=0. \tag{13}$$

The first-order conditions for w_{1t} and w_{2t} are, respectively,

$$H_{w_{1t}}=0 \qquad \text{and} \qquad H_{w_{2t}}=0.$$

We postulate $I(x_t,t)$ in a time-separable form as

$$I(x_t,t)=e^{-rt}[\beta_1\log(x_t)+\beta_0],$$

where β_0 and β_1 are to be determined from (13). By substituting the above candidate in (13), we obtain

$$\max_{w_{1t},w_{2t}} H(w_{1t},w_{2t};x_t,t) \equiv \max_{w_{1t},w_{2t}} \Bigg\{ \log(\psi x_t w_{1t}) \\ + \beta_1[r+(\gamma-r)w_{1t}+(\phi_V-r)w_{2t}] \\ - r[\beta_1\log(x_t)+\beta_0] \\ - \tfrac{1}{2}\beta_1(w_{1t}\sigma+w_{2t}\sigma_V)^2 \\ + \lambda\beta_1\mathrm{E}_\xi\left[\log(w_{1t}\xi+w_{2t}\xi_V+1)\right]\Bigg\}=0.$$

If we now compute the first-order conditions, we find that the optimal values of w_{1t} and w_{2t} satisfy:

$$\frac{1}{\beta_1 w_{1t}}+\mathrm{E}_\xi\left[\frac{\lambda\xi}{w_{1t}\xi+w_{2t}\xi_V+1}\right]+\gamma-r=(w_{1t}\sigma+w_{2t}\sigma_V)\sigma$$

and

$$\mathrm{E}_\xi\left[\frac{\lambda\xi_V}{w_{1t}\xi+w_{2t}\xi_V+1}\right]+\phi_V-r=(w_{1t}\sigma+w_{2t}\sigma_V)\sigma_V.$$

So far we have not made any assumption on the parameter values. From now on, without loss of generality, we assume that $\gamma=\phi-r$, that is, $r=\psi$.

4 Pricing the real option of waiting when consumption can be delayed

If we suppose a corner solution, $w_{1t} = 1$ and $w_{2t} = 0$, then

$$\frac{1}{\beta_1} + \lambda \mathrm{E}_\xi \left[\frac{\xi}{\xi+1} \right] + \gamma - r = \sigma^2 \tag{14}$$

and

$$\lambda \mathrm{E}_\xi \left[\frac{\xi_V}{\xi+1} \right] + \phi_V - r = \sigma \sigma_V. \tag{15}$$

In this case, it can be shown that $\beta_1 = r^{-1}$. After some simple computations, we have that equations (14) and (15) collapse in

$$\phi = r + \sigma^2 - \lambda \mathrm{E}_\xi \left[\frac{\xi}{\xi+1} \right], \tag{16}$$

and

$$\lambda \mathrm{E}_\xi \left[\frac{\xi_V}{\xi+1} \right] + \phi_V - r = \sigma \sigma_V. \tag{17}$$

From (17), it follows

$$\begin{aligned} &\lambda \mathrm{E}_\xi \left[\frac{V(S_t(\xi+1),t) - V(S_t,t)}{\xi+1} \right] \\ &+ \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} \phi S_t + \tfrac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 \right) - rV = \frac{\partial V}{\partial S_t} \sigma^2 S_t. \end{aligned}$$

If we now substitute (16) in the above equation, we get

$$\begin{aligned} &\lambda \mathrm{E}_\xi \left[\frac{V(S_t(\xi+1),t) - V(S_t,t) - \xi S_t \frac{\partial V}{\partial S_t}}{\xi+1} \right] \\ &+ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S_t} r S_t + \tfrac{1}{2} \frac{\partial^2 V}{\partial S_t^2} \sigma^2 S_t^2 - rV = 0. \end{aligned} \tag{18}$$

We impose the boundary conditions $V(0,t) = 0$ and $V(S_t,T) = \max(S_t - K, 0)$ where K is the exercise price of the real option (the cost, in terms of goods, of delaying consumption until the "last minute" $= T$). In such a case, without loss of generality, we may consider a finite planning horizon $[0,T]$ in the expected utility expressed in (12). Notice that if $f_\xi(\cdot)$

is the density function of ξ, then the presence of the expected value in the above equation given by

$$\begin{aligned}&\mathrm{E}_\xi\left[\frac{V(S_t(1+\xi),t)-\lambda V(S_t,t)}{\xi+1}\right]\\&=\int_{-\infty}^{\infty}\frac{V(S_t(1+\xi),t)-\lambda V(S_t,t)}{\xi+1}f_\xi(\xi)\mathrm{d}\xi\end{aligned}$$

produces in (18) a (partial) differential-integral equation. Notice that if ξ is constant in (18), by redefining λ as $\lambda/(\xi+1)$, we obtain Merton's (1976) formula. Finally, observe that when $\xi=0$ or $\lambda=0$, equation (18) reduces to the Black-Scholes' (1973) second order parabolic partial differential equation. Observe now that if we introduce the following change of variable:

$$\zeta=\left(\frac{y-\nu}{\kappa}\right)^{-\alpha},$$

then one of the expectations terms in (18) satisfies

$$\begin{aligned}\mathrm{E}\left[\frac{\xi}{\xi+1}\right]&=\mathrm{E}\left[\frac{X^{-\alpha}}{X^{-\alpha}-1}\right]\\&=\int_0^{\infty}\frac{[(y-\nu)/\kappa]^{-\alpha}}{[(y-\nu)/\kappa]^{-\alpha}-1}f_Y(y)\mathrm{d}y\\&=\int_0^{\infty}\frac{\zeta}{\zeta-1}e^{-\zeta}\mathrm{d}\zeta\\&=-e\Gamma(-1,1),\end{aligned}$$

where $\Gamma(-1,1)=-\Gamma(0,1)+e^{-1}$, $\Gamma(0,0)=\infty$, $\Gamma(0,\infty)=0$, and $\Gamma(0,1)\approx 2/9$ (in fact, $\Gamma(0,1)=0.219383934...$). Here, $\Gamma(a,b)$ denotes the incomplete Gamma function. In such a case, equation (18) can be transformed into

$$\begin{aligned}&\lambda\mathrm{E}_\xi\left[\frac{V(S_t(1+\xi),t)-V(S_t,t)}{\xi+1}\right]+\frac{\partial V}{\partial t}\\&+\tfrac{1}{2}\sigma^2S_t^2\frac{\partial^2 V}{\partial S_t^2}+\left[r+\lambda(\tfrac{2}{9}e-1)\right]S_t\frac{\partial V}{\partial S_t}-rV=0.\end{aligned}$$

A possibility to determine $V(S_t,t)$ consists in defining a sequence of random variables Y_n, each defined as the product of n independent and identically distributed random variables $\xi+1$, with $Y_0=1$. In other words, if $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables. We define

$$\begin{aligned}
Y_0 &= 1\\
Y_1 &= \xi_1 + 1\\
Y_2 &= (\xi_1 + 1)(\xi_2 + 1)\\
&\vdots\\
Y_n &= \prod_{k=1}^{n}(\xi_k + 1)\\
&\vdots
\end{aligned}$$

In this case, the solution of equation (18) with the boundary conditions

$$V(0,t) = 0, \qquad \text{and} \qquad V(S_t,T) = \max(S_t - K, 0),$$

is given by

$$V(S_t,t) = \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n}\Bigg[\frac{e^{-\lambda(T-t)/(\xi+1)}[\lambda(T-t)/(\xi+1)]^n}{n!} \times V_{\text{BS}}(S_t Y_n e^{-\lambda \mathrm{E}_\xi[\xi/(\xi+1)](T-t)}, t)\Bigg], \tag{19}$$

where ξ is independent of $\{\xi_n\}_{n\in\mathbb{N}}$ and $V_{\text{BS}}(\cdot,\cdot)$ is the basic Black-Scholes solution. Indeed, consider

$$V(S_t,t) = \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n}[P_{n,t}\, V_{\text{BS}}^{(n)}\,], \tag{20}$$

where

$$P_{n,t} = \frac{e^{-\lambda(T-t)/(\xi+1)}[\lambda(T-t)/(\xi+1)]^n}{n!},$$

$$U_{n,t} = Y_n e^{-\lambda \mathrm{E}_\xi[\xi/(\xi+1)](T-t)}$$

and

$$V_{\text{BS}}^{(n)} = V_{\text{BS}}(S_t U_{n,t}, t).$$

In what follows, it will convenient to introduce the notation

$$Q_{n,t} = S_t U_{n,t}.$$

In such a case,

$$\frac{\partial V}{\partial S_t} = \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; U_{n,t} \frac{\partial V^{(n)}_{\mathrm{BS}}}{\partial Q_{n,t}} \right], \tag{21}$$

$$\frac{\partial^2 V}{\partial S_t^2} = \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; U_{n,t}^2 \frac{\partial^2 V^{(n)}_{\mathrm{BS}}}{\partial Q_{n,t}^2} \right] \tag{22}$$

and

$$\begin{aligned} \frac{\partial V}{\partial t} =& \lambda \mathrm{E}_\xi[\xi/(\xi+1)] \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; Q_{n,t} \frac{\partial V^{(n)}_{\mathrm{BS}}}{\partial Q_{n,t}} \right] \\ &+ \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; \frac{\partial V^{(n)}_{\mathrm{BS}}}{\partial t} \right] \\ &+ \lambda \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[\frac{P_{n,t}\; V^{(n)}_{\mathrm{BS}}}{\xi+1} \right] \\ &- \lambda \sum_{n=1}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[\frac{e^{-\frac{\lambda(T-t)}{\xi+1}} \left[\frac{\lambda(T-t)}{\xi+1} \right]^{n-1}}{(n-1)!} \left(\frac{V^{(n)}_{\mathrm{BS}}}{\xi+1} \right) \right]. \end{aligned} \tag{23}$$

Hence, by virtue of (22) and (23), we get

$$\begin{aligned} \frac{\partial V}{\partial t} =& \lambda \mathrm{E}_\xi[\xi/(\xi+1)] S_t \frac{\partial V}{\partial S_t} \\ &+ \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; \frac{\partial V^{(n)}_{\mathrm{BS}}}{\partial t} \right] + \lambda \mathrm{E}_\xi \left[\frac{V(S_t,t)}{\xi+1} \right] \\ &- \lambda \sum_{m=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_{m+1}} \left[\frac{e^{-\frac{\lambda(T-t)}{\xi+1}} \left[\frac{\lambda(T-t)}{\xi+1} \right]^{m}}{m!} \left(\frac{V^{(m+1)}_{\mathrm{BS}}}{\xi+1} \right) \right]. \end{aligned} \tag{24}$$

Observe that the last term in the above equation can be written as

$$\begin{aligned} \mathrm{E}_\xi \left[\frac{V((\xi+1)S_t,t)}{\xi+1} \right] &= \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_n} \left[P_{n,t}\; \frac{V^{(n)}_{\mathrm{BS}}(Q_{n,t}(1+\xi),t)}{\xi+1} \right] \\ &= \sum_{n=0}^{\infty} \mathrm{E}_\xi \mathrm{E}_{Y_{n+1}} \left[P_{n,t}\; \frac{V^{(n+1)}_{\mathrm{BS}}(Q_{n+1,t},t)}{\xi+1} \right] \end{aligned} \tag{25}$$

since $Q_{n+1,t}$ y $Q_{n,t}(\xi+1)$ are independent and identically distributed random variables. Therefore, equation (24) is transformed into

$$\begin{aligned}\frac{\partial V}{\partial t} &= \sum_{n=0}^{\infty} \mathrm{E}_{\xi}\mathrm{E}_{Y_n}\left[P_{n,t}\,\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t}\right] \\ &\quad - \lambda \mathrm{E}_{\xi}\left[\frac{V(S_t(\xi+1),t) - V(S_t,t) - \xi S_t \frac{\partial V}{\partial S_t}}{\xi+1}\right]. \end{aligned} \tag{26}$$

From (21), (22) and (26), it follows that

$$\begin{aligned} &\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} + rS_t\frac{\partial V}{\partial S_t} - rV \\ &= \sum_{n=0}^{\infty} P_{n,t}\mathrm{E}_{\xi}\mathrm{E}_{Y_n}\left[\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t} + \tfrac{1}{2}\sigma^2 Q_{n,t}^2\frac{\partial^2 V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}^2} + rQ_{n,t}\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}} - rV_{\mathrm{BS}}^{(n)}\right] \\ &\quad - \lambda \mathrm{E}_{\xi}\left[\frac{V(S_t(\xi+1),t) - V(S_t,t) - \xi S_t \frac{\partial V}{\partial S_t}}{\xi+1}\right]. \end{aligned} \tag{27}$$

Since

$$\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial t} + \tfrac{1}{2}\sigma^2 Q_{n,t}^2\frac{\partial^2 V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}^2} + rQ_{n,t}\frac{\partial V_{\mathrm{BS}}^{(n)}}{\partial Q_{n,t}} - rV_{\mathrm{BS}}^{(n)} = 0$$

holds for all $n \in \mathbb{N}\cup\{0\}$, we deduce, immediately, that (19) is solution of (18).

5 Numerical approximations

In order to obtain numerical approximations of (19), the quantity inside the mathematical expectations in (19)

$$M_{\xi,Y_n} = \sum_{n=0}^{1000} \frac{e^{-\lambda(T-t)/(\xi+1)}[\lambda(T-t)/(\xi+1)]^n}{n!} V_{\mathrm{BS}}^{(n)} \tag{28}$$

is simulated by using the statistical software "Xtremes" (Reiss and Thomas, 2001) and Ripley's methodology (1987) for Monte Carlo simulations. Subsequently, we compute the average of 10,000 simulated values of M_{ξ,Y_n} to obtain, for different values of λ, approximate solutions of the real option of waiting when consumption can be delayed. To do this, let us first consider the following parameter values, in Table 1, for computing the basic Black-Scholes price $V_{\text{BS}}^{(0)}$. In Table 1, S_t stands for the price of money in terms of goods, K is the cost (in terms of goods) of delaying consumption until the last minute, r is the nominal interest rate, and $T-t$ is the term. Units of S_t and K are given in money in terms of consumption goods.

Parameters for Black-Scholes price of the real option					
S_t	K	r	σ	$T-t$	$V_{\text{BS}}^{(0)}$
42.00	41.00	0.11	0.13	0.25	2.436

Table 1: Parameter values of the benchmark Black-Scholes price.

Table 2 shows numerical approximation of the price of the real option by using Monte Carlo simulation for different values of λ with $\text{E}_\xi[\xi/(\xi+1)] = -e\Gamma(-1,1)$. It is assumed, for simulation purposes, that ξ follows a Fréchet distribution with mean 0.01 and variance 0.001.

Real option price					
$\text{E}_\xi[\xi/(\xi+1)] = -e\Gamma(-1,1)$					
λ	0.1	0.2	0.3	0.4	0.5
$V(S_t,t)$	2.646	2.673	2.698	2.726	2.742
(*Cont.*)					
λ	0.6	0.7	0.8	0.9	1.0
$V(S_t,t)$	2.845	2.865	2.898	3.012	3.081

Table 2: Simulated prices of the real option.

It is important to point out that option prices in Table 2 depend of the choices of the mean and variance of the random variable ξ. We may conclude, from Table 2 and the chosen mean and variance, that the price of the real option of waiting when consumption can be delayed increases when the average number of jumps per unit of time increases since a growing λ rises the future opportunity cost of purchasing goods.

6 Conclusions

We have developed a stochastic model of a small open monetary economy in which agents have expectations of the exchange-rate dynamics guided by a mixed diffusion-jump process. The size of a possible exchange-rate depreciation is supposed to have an extreme value distribution of the Fréchet type. By using a logarithmic utility, we have derived an analytical solution for valuing the real option of waiting when consumption can be delayed; a claim that is not traded. The provided explicit solutions have made much easier the understanding of the key issues of extreme jumps in valuing contingent claims in a cash-in-advance economy. Finally, a Monte Carlo simulation was carried out to obtain approximate solutions of the real option price.

It is worthwhile mentioning that the derived results do not depend on the assumption of logarithmic utility, which is a limit case of the family of constant relative risk aversion utility functions. Needles to say, both nontradable and durable goods will provide more realistic assumptions and should be considered in extending, in further research, the real option of waiting when consumption can be delayed.

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