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Shallow potential wells for the Schrödinger equation and water waves *

Peter Zhevandrov [1] Anatoli Merzon

Abstract

We propose a simple method for constructing asymptotics of eigenfunctions for the Schrödinger equation with a shallow potential well and its generalization to the problem of water waves trapped by an underwater ridge.

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1 Introduction

It is well-known that the Schrödinger equation

$$(-\Delta + U)\Psi = E\Psi \tag{1.1}$$

in the case when U describes a shallow potential well (i.e., $U = \varepsilon V(x)$, $V(x) \in C_0^\infty(\mathbf{R}^n)$, $\varepsilon \to 0$) has exactly one eigenvalue $E_0 = -\beta^2$, $\beta \in \mathbf{R}$, below the essential spectrum $[0, \infty)$ in the case when $\int_{\mathbf{R}^n} V(x)dx \leq 0$ and the dimension n of the configuration space is 1 or 2. This was established for $n = 1$ and in the radially symmetric case for $n = 2$ already in the famous textbook of Landau&Lifshitz [5] and later was demonstrated in the general case in dimension 2 by Simon [7]. The methods used by those authors are quite different and consist, in brief, in the following. Landau&Lifshitz construct the asymptotics of the eigenfunction in the

*Invited article.

[1] Institute of Mathematics, UNAM (campus Morelia); on leave from Institute of Physics and Mathematics, University of Michoacán, Morelia, Mich., MEXICO.

domains where $V \equiv 0$ and $V \not\equiv 0$ separately and then glue them together; thus, the asymptotics of the eigenfunction is nonuniform and the method *per se* is applicable only in the radially symmetric case for $n = 2$. The asymptotics of the eigenvalues is obtained from the gluing conditions. On the other hand, Simon reduces the problem to an equation for the eigenvalues (secular equation) which he solves by means of a Taylor expansion using the implicit function theorem; thus in his approach the asymptotics of the eigenfunction does not appear at all. Moreover, Simon's method is by no means trivial because it uses, for example, the theory of nuclear operators. Close results on the limiting behavior of the resolvent can be found in [1].

Our goal here is to construct a uniform asymptotics of the eigenfunction in this situation assuming that

$$C_1 \geq \|\Psi\| \geq C_2 > 0 \quad \text{as} \quad \varepsilon \to +0, \tag{1.2}$$

where $C_{1,2}$ do not depend on ε and the norm is that of $L_2(\mathbf{R})$. It turns out that this construction is completely elementary when one passes to the momentum representation. Moreover, our method is equally efficient for the Schrödinger equation and the problem of water waves trapped by a submarine ridge, which, as it is known in the folklore, is analogous to the Schrödinger equation with a potential well. The corresponding problem after the passage to dimensionless variables reads as follows:

$$\begin{gathered} \Delta\Phi - \Phi = 0, \quad -h(x) < y < 0, \\ \partial\phi/\partial n = 0, \quad y = -h(x), \\ \Phi_y = \omega^2\Phi, \quad y = 0; \end{gathered} \tag{1.3}$$

here $\Phi \in H_1(-h < y < 0, x \in \mathbf{R})$ is the velocity potential, $h(x)$ is the depth, x and y are the horizontal and vertical coordinates, respectively, and ω is the frequency and at the same time the spectral parameter. We assume that $h(x) = h_0 + \varepsilon V(x), V(x) \in C_0^\infty(\mathbf{R})$. From the results of [2] it follows that for sufficiently small ε there exists exactly one eigenvalue ω^2 below the essential spectrum $[\tanh h_0, \infty)$ when $\int_{\mathbf{R}} V(x) \leq 0$. The asymptotics of this eigenvalue was obtained in [4] for the strict inequality in the last formula and in a closely related but different asymptotic regime (long-wave approximation).

We prove the following theorems. Denote

$$\tilde{V}(p) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ipx} V(x) dx.$$

Theorem 1.1 *(Schrödinger equation, the case of dimension 1)*

1) Let

$$\int_{\mathbf{R}} V(x)dx < 0. \tag{1.4}$$

Then

$$\Psi_n(x) = \mu_n^{3/2} \int_{\mathbf{R}} e^{ipx} \frac{a_0(p) + \varepsilon a_1(p) + \ldots + \varepsilon^n a_n(p)}{p^2 + \mu_n^2} dp, \tag{1.5}$$

$n = 0, 1, 2, \ldots$, *is the asymptotics of the eigenfunction* Ψ *satisfying condition (1.2) and belonging to the eigenvalue*

$$E = -\mu_n^2 + O(\varepsilon^{n+5/2}), \tag{1.6}$$

i.e.

$$\|\Psi - \Psi_n\| = O(\varepsilon^{n+1/2}) \quad \text{as} \quad \varepsilon \to +0. \tag{1.7}$$

Moreover,

$$\mu_n = \varepsilon(\beta_0 + \varepsilon\beta_1 + \ldots + \varepsilon^n \beta_n), \quad \beta_0 = -\sqrt{\frac{\pi}{2}} \tilde{V}(0), \quad a_0(p) = \frac{\tilde{V}(p)}{\tilde{V}(0)},$$

and the remaining values $\beta_1, \ldots, \beta_n$ *and functions* $a_1, \ldots, a_n$ *are determined from system (3.10-3.13)*

2) Let $\int_{\mathbf{R}} V(x)dx = 0$. *Then*

$$\Psi_n(x) = \mu_n \int_{\mathbf{R}} e^{ipx} \frac{a_0(p) + \varepsilon a_1(p) + \ldots + \varepsilon^n a_n(p)}{p^2 + \mu_n^2} dp, \tag{1.8}$$

$n = 1, 2, \ldots$, *is the asymptotics of the eigenfunction* Ψ *satisfying condition (1.2) and belonging to the eigenvalue* $E = -\mu_n^2 + O(\varepsilon^{n+4})$ *in the sense that*

$$\|\Psi - \Psi_n\| = O(\varepsilon^n). \tag{1.9}$$

Moreover,

$$\mu_n = \varepsilon^2(\beta_1 + \varepsilon\beta_2 + \ldots + \varepsilon^n \beta_n),$$
$$\beta_1 = \frac{1}{2} \int_{\mathbf{R}} \frac{|V(t)|^2}{t^2} dt,$$
$$\beta_2 = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbf{R}+i} \int_{\mathbf{R}+i} \frac{\tilde{V}(t-s)\tilde{V}(s)\tilde{V}(-t)}{t^2 s^2} dt ds,$$
$$a_0(p) = \tilde{V}(p),$$
$$a_1(p) = -\beta_2 \tilde{V}(p) - i\sqrt{\frac{\pi}{2}} \tilde{V}(p)\tilde{V}'(0) - \sqrt{\frac{2}{\pi}} \int_{\mathbf{R}+i} \frac{\tilde{V}(p-t)\tilde{V}(t)}{t^2} dt,$$

and the remaining values of $\beta_3, \ldots, \beta_n$ *and the functions* $a_2, \ldots, a_n$ *are determined from system (3.10-3.13).*

Theorem 1.2 *(Schrödinger equation, the case of dimension 2)*

1) Let $\int_{\mathbf{R}^2} V(x)dx < 0$. Then

$$\Psi_0(x) = \mu_1 \int_{\mathbf{R}^2} e^{ipx} \frac{a_0(p)}{p^2+\mu_1^2} dp,$$

where

$$\begin{gathered} \mu_1 = \exp \frac{1}{\varepsilon(\alpha_0 + \varepsilon\alpha_1)}, \\ \alpha_0 = \tilde{V}(0), \quad \alpha_1 = -\frac{1}{2\pi} ⨍_{\mathbf{R}^2} \frac{\tilde{V}(p)\tilde{V}(-p)}{p^2} dp, \\ a_0(p) = \tilde{V}(p), \end{gathered} \tag{1.10}$$

is the asymptotics of the eigenfunction belonging to the eigenvalue $E = -\mu_1^2 + O(\mu_1^2\varepsilon^2))$ in the sense

$$\|\Psi - \Psi_1\| = O(\varepsilon). \tag{1.11}$$

Here $⨍ \frac{f(p)}{p^2} dp = \int_{|p|<1} \frac{f(p)-f(0)}{p^2} dp + \int_{|p|>1} \frac{f(p)}{p^2} dp.$

2) Let $\int_{\mathbf{R}^2} V(x)dx = 0$. Then

$$\Psi_1(x) = \frac{\mu_3}{\varepsilon} \int_{\mathbf{R}^2} e^{ipx} \frac{a_0(p) + \varepsilon a_1(p)}{p^2 + \mu_3^2} dp, \tag{1.12}$$

where

$$\mu_3 = \exp \frac{1}{\varepsilon(\varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3)},$$

α_1, a_0 are determined from (1.10),

$$\begin{aligned} \alpha_2 &= \frac{\alpha_1}{(2\pi)^2} \int_{\mathbf{R}^4} \frac{\tilde{V}(-p)\tilde{V}(p-t)\tilde{V}(-p)a_0(t)}{p^2t^2} dpdt, \\ \alpha_3 &= \frac{\alpha_2}{(2\pi)^2} \int_{\mathbf{R}^4} \frac{\tilde{V}(-p)\tilde{V}(p-t)\tilde{V}(-p)a_0(t)}{p^2t^2} dpdt \\ &\quad + \frac{\alpha_1}{(2\pi)^2} \int_{\mathbf{R}^2} \frac{\tilde{V}(p)}{p^2} \left(⨍_{\mathbf{R}^2} \frac{\tilde{V}(p-s)a_1(s)}{s^2} ds \right) dp, \\ a_1(p) &= -\frac{\alpha_1}{2\pi} \int_{\mathbf{R}^2} \frac{\tilde{V}(p-t)a_0(t)}{t^2} dt - \alpha_2 a_0(p), \end{aligned}$$

is the asymptotics of the eigenfunction belonging to the eigenvalue $E = -\mu_3^2 + O(\mu_3^2\varepsilon)$ in the sense $\|\Psi - \Psi_1\| = O(\varepsilon)$.

Remark. It is possible to construct corrections of any order to the asymptotic eigenfunction Ψ_1.

Theorem 1.3 *1) Let $\int_{\mathbf{R}} V(x)dx < 0$. Then*

$$\Psi_n(x) = \mu_n^{3/2} \int_{\mathbf{R}} e^{ipx} \frac{a_0(p) + \varepsilon a_1(p) + \ldots + \varepsilon^n a_n(p)}{\cosh h_0 [L(p) + \mu_n^2]} dp, \tag{1.13}$$
$$L(p) = \sqrt{1+p^2}\tanh(\sqrt{1+p^2}h_0) - \tanh h_0,$$

is the asymptotics of the trapped wave Ψ satisfying condition (1.2) and corresponding to the frequency

$$\omega^2 = \tanh h_0 - \mu_n^2 + O(\varepsilon^{n+5/2}) \tag{1.14}$$

in the sense (1.7). Moreover,

$$\mu_n = \varepsilon(\beta_0 + \varepsilon\beta_1 + \ldots + \varepsilon^n \beta_n),$$
$$\beta_0 = \frac{1}{\sqrt{2l}\cosh^2 h_0} \int_{\mathbf{R}} V(x)dx, \quad a_0(p) = \frac{\tilde{V}(p)}{\tilde{V}(0)}, \tag{1.15}$$
$$\text{where} \quad l = \tanh h_0 - h_0(\tanh h_0)^2 + h_0,$$

and the remaining values $\beta_1, \ldots, \beta_n$ and functions $a_1, \ldots, a_n$ are determined from the corresponding system.

2) Let $\int_{\mathbf{R}} V(x)dx = 0$. Then

$$\Psi_n(x) = \mu_n \int_{\mathbf{R}} e^{ipx} \frac{a_0(p) + \varepsilon a_1(p) + \ldots + \varepsilon^n a_n(p)}{\cosh h_0 [L(p) + \mu_n^2]} dp \tag{1.16}$$

is the asymptotics of the trapped wave Ψ satisfying condition (1.2) and corresponding to the frequency

$$\omega^2 = \tanh h_0 - \mu_n^2 + O(\varepsilon^{n+4}) \tag{1.17}$$

in the sense (1.9). Moreover,

$$\mu_n = \varepsilon^2(\beta_1 + \varepsilon\beta_2 + \ldots + \varepsilon^n \beta_n),$$
$$\beta_1 = \frac{1}{\sqrt{2l}\cosh^2 h_0} \int_{\mathbf{R}} |\tilde{V}(p)|^2 f(p)dp, \quad a_0(p) = \frac{\sqrt{\pi}\tilde{V}(p)}{\beta_1\sqrt{l}\cosh^2 h_0},$$

where $f(p)$ is a positive function, $f(p) = \dfrac{\tau^2 - \tau\tanh h_0 \tanh(\tau h_0)}{\tau \tanh \tau h_0 - \tanh h_0}$, $\tau = \sqrt{1+p^2}$, l is defined from (1.15) and the remaining values $\beta_2, \ldots, \beta_n$ and functions $a_1, \ldots, a_n$ are determined from the corresponding system.

2 Heuristic considerations

Before we go further, we would like to give some heuristic considerations which explain the specific form of the asymptotics of the eigenfunctions in Theorems 1.1-1.3.

We will present these arguments only in the simplest case of the Schrödinger equation in dimension 1; their generalizations for other cases are straightforward. Thus we would like to construct an approximate solution of

$$-\Psi'' + \varepsilon V(x)\Psi = E\Psi. \tag{2.1}$$

We already know (although we will also obtain this fact) that in the case $\int V(x)dx < 0$ the energy $E = O(\varepsilon^2)$, $E = -\mu^2$, say. After performing the Fourier transform in (2.1) we obtain

$$\tilde{\Psi}(p)(p^2 + \mu^2) = -\varepsilon \int \tilde{V}(p - p')\tilde{\Psi}(p')dp', \tag{2.2}$$

where the tilde denotes the Fourier transform. Obviously, for $x \notin \operatorname{supp} \ V(x)$ we have $\Psi \sim e^{-\mu|x|}$ with appropriate constants and the Fourier transform of this function is a δ-type sequence. Hence the integral in the RHS of (2.2) is approximately equal to

$$-\varepsilon C\tilde{V}(p)$$

with some normalization constant C. Therefore, by (2.2), $\tilde{\Psi}(p)$ is approximately equal to

$$\text{Const}\, \frac{A(p)}{p^2 + \mu^2}, \tag{2.3}$$

where $A(p) = \tilde{V}(p)$ is a function from $S(\mathbf{R})$. The singular dependence of the eigenfunction on ε is reflected in the denominator in (2.3). We note that the structure of (2.3) is identical to formulas of Theorems 1.1-1.3. Further, expanding A and μ in regular series in ε, calculating the asymptotics of the integral in (2.2) and equating to zero the coefficients of like powers of ε, one obtains the complete asymptotic series for Ψ. The theorem on closeness of formal asymptotics to the exact solution [6] provides the final step of the proof.

In conclusion, a few words about the water wave problem. As shown in [9], problem (1.3) reduces to the following integral equation for the function $\phi(x) = \Phi|_{y=0}$:

$$\tilde{\phi}(p)(L(p) + \mu^2) = \varepsilon \int_\gamma M(\epsilon, p, p')\tilde{\phi}(p')dp', \tag{2.4}$$

where $\mu^2 = \tanh h_0 - \omega^2$, $L(p)$ is defined in (1.13), γ is an appropriate contour in the complex plane, and the function $M(\varepsilon, p, p')$ is analytic in p, p' along γ and linear in ε. Since $L(p) \sim \mathrm{Const}\, p^2$ for small p, we see that (2.4) is similar to (2.2) and our arguments are still valid if we change the denominator in (2.3) to $L(p) + \mu^2$.

3 Sketch of the proof

In this section we will give a (rather detailed) sketch of the proof of the first item of Theorem 1.1; the idea of the proof of the other statements is similar.

Passing to the Fourier transform in (1.1) we obtain

$$(p^2 - E)\tilde{\Psi}(p) = -\frac{\varepsilon}{\sqrt{2\pi}} \int_{\mathbf{R}} \tilde{V}(p - p')\tilde{\Psi}(p')dp'. \tag{3.1}$$

According to the scheme outlined in the previous section, we look for the approximate solution of this equation in the form

$$\tilde{\Psi}_n(p) = \varepsilon B_n \frac{A_n(p)}{p^2 + \varepsilon^2 B_n^2}, \tag{3.2}$$

$$A_n(p) = a_0(p) + \varepsilon a_1(p) + \ldots + \varepsilon^n a_n(p).$$

We assume that $a_0(p) \not\equiv 0$ and

$$B_n = \beta_0 + \varepsilon\beta_1 + \ldots + \varepsilon^n \beta_n. \tag{3.3}$$

The approximate energy level is

$$E_n = -\varepsilon^2 B_n^2. \tag{3.4}$$

We will look for the solution satisfying the normalization conditions

$$a_0(0) = 1, \quad a_k(0) = 0, \quad k = 2, \ldots, n. \tag{3.5}$$

Our goal is to construct such values of $\beta_0, \beta_1 \ldots, \beta_n$ and functions $a_0(p), \ldots, a_n(p)$ that $\tilde{\Psi}_n(p)$ satisfy equation (3.1) up to $O(\varepsilon^{n+2})$, where $\|O(\varepsilon^{n+2})\| \leq \mathrm{Const}\, \varepsilon^{n+2}$.

Substituting (3.2) and (3.4) in (3.1) we obtain an equivalent equation

$$\varepsilon B_n A_n(p) = -\frac{\varepsilon^2 B_n}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{\tilde{V}(p - p')A_n(p')dp'}{p'^2 + \varepsilon^2 B_n^2}. \tag{3.6}$$

We will need an auxiliary lemma on the asymptotics expansions of integrals of the form

$$\int_{\mathbf{R}} \frac{\phi(p,t)}{t^2+\mu^2}dt \tag{3.7}$$

as $\mu \to 0$, where $\phi(p,t)$ is an entire function in t and belongs to $S(\mathbf{R}_t)$ uniformly in $p \in \mathbf{R}$. There are several methods of calculating such asymptotics (see [3]); for our case the method based on the calculus of residues turns out to be more convenient. Introduce the contour in the complex plane $\mathbf{C}$:

$$\gamma_1 := (-\infty,-1] \cup \{x+iy : x^2+y^2=1, |x| \le 1, y>0\} \cup [1,+\infty).$$

Lemma 3.1 *Let $\phi(t)$ be an entire function and $\phi(t) \in S(\mathbf{R}), t \in \mathbf{R}$. Then as $\mu \to 0$*

$$\begin{aligned}\int_{\mathbf{R}} \frac{\phi(t)dt}{t^2+\mu^2} \quad = \quad & \sum_{k=0}^{n} \alpha_k \mu^k \int_{\gamma_1} \frac{\phi(t)dt}{t^{k+2}} \\ & +\mu^{2([\frac{n}{2}]+1)} \int_{\gamma_1} \frac{\phi(t)dt}{t^{2([\frac{n}{2}]+1)}(t^2-\mu^2)} \\ & +\frac{\pi}{\mu}\{\sum_{k=0}^{n}(i\mu)^k \frac{\phi^k(0)}{k!} \\ & +\frac{(i\mu)^{n+1}}{n!}\int_0^1 (1-t)^n \phi^{(n+1)}(ti\mu)dt\},\end{aligned} \tag{3.8}$$

where $\alpha_k = (1+(-1)^k)/2$.

We do not give the proof of this lemma.

Let us continue the proof of Theorem 1.1. Expanding the left hand side of (3.6) in ε, using (3.2) and (3.3), we obtain

$$\varepsilon B_n A_n(p) = \sum_{k=1}^{n+1} \varepsilon^k (\sum_{l=0}^{k} \beta_l a_{k-l}(p)) + \varepsilon^{n+2} R_{n+2}(\varepsilon,\bar{\beta},\bar{a}), \tag{3.9}$$

where $\bar{\beta} = (\beta_0,\beta_1,\ldots,\beta_n)$, $\bar{a} = (a_0,a_1,\ldots,a_n)$ and $R_{n+2}(\cdot,\cdot,\cdot)$ is a polynomial in its arguments. Substituting in Lemma 3.1 $\mu = \varepsilon B_n$, $\phi(t) = \varepsilon B_n \tilde{V}(p-t)A_n(t)$ and calculating the coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2$ and also observing how $\beta_0,\ldots,\beta_{n-1}, a_{n-2}, a_{n-1}, a_n$ enter the coefficient of ε^n, we obtain the expansion of the integral in the right hand side of equation (3.6):

$$
\begin{aligned}
&\varepsilon B_n \int_{\mathbf{R}} \frac{\tilde V(p-t)A_n(t)dt}{t^2+\varepsilon^2 B_n^2} \\
&\quad = \pi a_0(0)\tilde V(p) \\
&\qquad -\varepsilon\pi\{\beta_0[ia_0(0)\tilde V'(p) - ia_0'(0)\tilde V(p) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt] \\
&\qquad\quad -a_1(0)\tilde V(p)\} \\
&\qquad -\varepsilon^2\pi\{\beta_0[ia_1(0)\tilde V'(p) - ia_1'(0)\tilde V(p)) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_1(t)dt] \\
&\qquad\quad +\beta_0^2[\frac{1}{2}a_0(0)\tilde V''(p) - a_0'(0)\tilde V'(p) + \frac{1}{2}a_0''(0)\tilde V(p)] \\
&\qquad\quad +\beta_1[ia_0(0)\tilde V'(p) - ia_0'(0)\tilde V(p) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt] \\
&\qquad\quad -a_2(0)\tilde V(p)\} + \dots \\
&\qquad -\varepsilon^n\pi\{\beta_0[ia_{n-1}(0)\tilde V'(p) - ia_{n-1}'(0)\tilde V(p) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_{n-1}(t)dt] \\
&\qquad\quad +\beta_0^2[\frac{1}{2}a_{n-2}(0)\tilde V''(p) - a_{n-2}'(0)\tilde V'(p) + \frac{1}{2}a_{n-2}''(0)\tilde V(p)] \\
&\qquad\quad +\beta_1[ia_{n-2}(0)\tilde V'(p) - ia_{n-2}'(0)\tilde V(p) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_{n-2}(t)dt] \\
&\qquad\quad + \dots \\
&\qquad\quad +\beta_{n-1}[ia_0(0)\tilde V'(p) - ia_0'(0)\tilde V(p) - \frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt] \\
&\qquad\quad -a_n(0)\tilde V(p)\} \\
&\qquad +\varepsilon^{n+1}S_{n+1}(p,\varepsilon,\bar\beta,\bar a),
\end{aligned}
$$

where

$$
\begin{aligned}
S_{n+1}(p,\varepsilon,\bar\beta,\bar a) &= \int_{\gamma_1}\frac{\tilde V(p-t)A_n(t)P_{n+1}(t,\varepsilon,\bar\beta)}{t^{m(n)}(t^2-\varepsilon^2B_n^2)}dt \\
&+ \int_0^1 Q_{n+1}(t,\varepsilon,\bar\beta)\ \frac{\partial^{n+1}}{\partial\tau^{n+1}}\left(\tilde V(p-\tau)A_n(\tau)\right)\Big|_{\tau=it\varepsilon B_n} dt,
\end{aligned}
$$

$P_{n+1}(\cdot,\cdot,\cdot)$, $\quad Q_{n+1}(\cdot,\cdot,\cdot)$ are polynomials in their arguments, $m(n)\in\mathbf{N}$ and $m(n)\to\infty$ as $n\to\infty$.

Multiplying the obtained expression by $-\varepsilon/\sqrt{2\pi}$ and equating in (3.6) the coefficients of ε^k, $k=1,\dots,n+1$, to zero, we obtain the system for β_k, a_k, $k=0,1,\dots,n$:

$$\beta_0 a_0(p) = -\sqrt{\frac{\pi}{2}}a_0(0)\tilde V(p), \tag{3.10}$$

$$\begin{aligned}&\beta_0 a_1(p)+\beta_1 a_0(p)\\ &=\sqrt{\frac{\pi}{2}}\{\beta_0[ia_0(0)\tilde V'(p)-ia_0'(0)\tilde V(p)-\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt]\\ &\quad -a_1(0)\tilde V(p)\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\beta_0 a_2(p)+\beta_1 a_1(p)+\beta_2 a_0(p)\\ &=\sqrt{\frac{\pi}{2}}\{\beta_0[ia_1(0)\tilde V'(p)-ia_1'(0)\tilde V(p))-\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_1(t)dt]\\ &\quad +\beta_0^2[\frac{1}{2}a_0(0)\tilde V''(p)-a_0'(0)\tilde V'(p)+\frac{1}{2}a_0''(0)\tilde V(p)]\\ &\quad +\beta_1[ia_0(0)\tilde V'(p)-ia_0'(0)\tilde V(p)-\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt]\\ &\quad -a_2(0)\tilde V(p)\},\end{aligned} \tag{3.12}$$

$$\dots$$

$$\begin{aligned}&\beta_0 a_n(p)+\beta_1 a_{n-1}(p)+\ldots+\beta_{n-1}a_1(p)+\beta_n a_0(p)\\ &=\sqrt{\frac{\pi}{2}}\{\beta_0[ia_{n-1}(0)\tilde V'(p)-ia_{n-1}'(0)\tilde V(p))\\ &\quad -\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_{n-1}(t)dt]\\ &\quad +\beta_0^2[\frac{1}{2}a_{n-2}(0)\tilde V''(p)-a_{n-2}'(0)\tilde V'(p)+\frac{1}{2}a_{n-2}''(0)\tilde V(p)]\\ &\quad +\beta_1[ia_{n-2}(0)\tilde V'(p)-ia_{n-2}'(0)\tilde V(p)\\ &\quad -\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_{n-2}(t)dt]+\ldots\\ &\quad +\beta_{n-1}[ia_0(0)\tilde V'(p)-ia_0'(0)\tilde V(p)-\frac{1}{\pi}\int_{\gamma_1}\frac{\tilde V(p-t)}{t^2}a_0(t)dt]\\ &\quad -a_n(0)\tilde V(p)\}.\end{aligned} \tag{3.13}$$

Lemma 3.2 *System (3.10)-(3.13) is uniquely solvable under conditions (3.5) and its solutions $a_0, \dots, a_n$ belong to $S(\mathbf{R})$.*

Proof: Setting $p = 0$ in (3.10) and taking into account that by (1.4) $\tilde{V}(0) \neq 0$ we obtain

$$\beta_0 = -\sqrt{\frac{\pi}{2}}\tilde{V}(0). \tag{3.14}$$

By the condition $a_0(0) = 1$ we obtain from (3.11)

$$a_0(p) = \frac{\tilde{V}(p)}{\tilde{V}(0)}. \tag{3.15}$$

Set $p = 0$ in (3.11). By (3.15) and the condition $a_1(0) = 0$ we obtain

$$\beta_1 = \frac{1}{2}\int_{\gamma_1} \frac{\tilde{V}(t)\tilde{V}(-t)}{t^2} dt. \tag{3.16}$$

Now let us find $a_1(p)$ from (3.11). Substituting (3.14), (3.15) and (3.16) in (3.11), and taking into account the fact that $a_0(0) = 1$, we obtain

$$\begin{aligned} a_1(p) \quad = \quad & \frac{i}{\tilde{V}(0)}\sqrt{\frac{\pi}{2}}[\tilde{V}'(p)\tilde{V}(0) - \tilde{V}'(0)\tilde{V}(p)] \\ & -\frac{1}{\sqrt{2\pi}\tilde{V}(0)}\int_{\gamma_1} \frac{\tilde{V}(t)\tilde{V}(p-t)}{t^2} dt \\ & +\frac{\tilde{V}(p)}{\sqrt{2\pi}\tilde{V}(0)^2}\int_{\gamma_1} \frac{\tilde{V}(t)\tilde{V}(-t)}{t^2} dt. \end{aligned} \tag{3.17}$$

We see that indeed $a_1(0) = 0$. Proceeding analogously, we obtain β_n and a_n assuming that $\beta_0, \beta_1, \dots, \beta_{n-1}$, $a_0, a_1, \dots, a_{n-1}$ are known and that $a_k(0) = 0, k = 2, \dots, n-1$. We look for $a_n(p)$ such that $a_n(0) = 0$. Setting $p = 0$ in (3.13) and taking into account the fact that $a_0(0) = 1$, we obtain

$$\begin{aligned} \beta_n \quad = \quad & \sqrt{\frac{\pi}{2}}\{\beta_0[-ia'_{n-1}(0)\tilde{V}(0) - \frac{1}{\pi}\int_{\gamma_1} \frac{\tilde{V}(-t)a_{n-1}(t)}{t^2} dt] + \dots \\ & +\beta_{n-1}[i\tilde{V}'(0) - ia'_0(0)\tilde{V}(0) - \frac{1}{\pi}\int_{\gamma_1} \frac{\tilde{V}(p-t)a_0(t)}{t^2} dt]\}; \end{aligned}$$

that is, β_n is uniquely determined. Substituting the last formula and $\beta_0, \dots, \beta_{n-1}$, $a_0, \dots, a_{n-1}$ in (3.13) we see that $a_n(p)$ is uniquely determined since $\beta_0 \neq 0$. It is not hard to see that indeed $a_n(0) = 0$.

Thus we are left with the proof of the fact that all the $a_k \in S(\mathbf{R})$. This is obvious for $k = 0$. For the function a_1 this follows from (3.17), the Peetre inequality [8]

$$(1 + |\theta|^2)^s \leq 2^{|s|}(1 + |\theta - \theta'|^2)^{|s|}(1 + |\theta'|^2)^s$$

for all $\theta, s \in \mathbf{R}$, and the condition $\tilde{V}(p-t) \in S(\mathbf{R}_+)$. The corresponding assertions for $a_k, k > 1$, are proved by induction. Lemma 3.2 is proved. $\square$

Let us complete the proof of Theorem 1.1. From Lemma 3.5 and (3.10-3.13) it follows that B_n and A_n expressed in terms of values $\beta_0, \ldots, \beta_n$ and functions $a_0 \ldots, a_n$ by means of (3.2), (3.3) solve equation (3.6) up to $O(\varepsilon^{n+2})$. This means that the function $\tilde{\Psi}_n(p)$ from (3.2) solves equation (3.1) up to $O(\varepsilon^{n+2})$. Using Lemma 1.3 from [6] we obtain after normalization (1.2) the estimate (1.6). Also from Lemma 1.4 of the same book we obtain the estimate (1.7) for the eigenfunction Ψ. The first item of Theorem 1.1 is proved. $\square$

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P. Zhevandrov
Institute of Mathematics,
UNAM (campus Morelia),
Morelia, Mich., MEXICO.
pzhevand@zeus.ccu.umich.mx

A. Merzon
Institute of Physics and Mathematics,
University of Michoacán,
Morelia, Mich., MEXICO.
anatoli@ginette.ifm.umich.mx

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