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The equations of the cone associated to the Rees algebra of the ideal of square-free k-products *

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Abstract

In this paper we determine the equations of the polyhedral cone generated by the exponent vectors of the monomials defining the Rees algebra of the ideal generated by the square-free monomials of degree k in n variables. Some applications are presented to show the relevance of the computation of these equations.

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1 The equations of the cone

In this paper we determine the equations of the polyhedral cone generated by the exponent vectors of the monomials defining the Rees algebra of the ideal generated by the square-free monomials of degree k in n variables (see Theorem 1.9). The importance of knowing those equations comes from the fact that the canonical module of the Rees algebra can be expressed in terms of the relative interior of the cone. This would allow to compute the a-invariant and the type of those Rees algebras. Another possible application is to find degree bounds for the generators of the integral closure of the Rees algebra of any ideal generated by square-free monomials of the same degree.

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Preliminaries on polyhedral geometry An *affine space* in $\mathbb{R}^n$ is by definition a translation of a linear subspace of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$ and $\mathrm{aff}(A)$ the affine space generated by A. Recall that $\mathrm{aff}(A)$ is the set of all *affine combinations* of points in A:

$$\mathrm{aff}(A) = \{a_1 p_1 + \cdots + a_r p_r \,|\, p_i \in A,\, a_1 + \cdots + a_r = 1,\, a_i \in \mathbb{R}\}.$$

There is a unique linear subspace V of $\mathbb{R}^n$ such that

$$\mathrm{aff}(A) = x_0 + V,$$

for some $x_0 \in \mathbb{R}^n$. The *dimension* of A is defined as $\dim A = \dim_{\mathbb{R}}(V)$.

If $0 \neq a \in \mathbb{R}^n$, then H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle = 0\},$$

where $\langle\, , \,\rangle$ is the usual inner product in $\mathbb{R}^n$. The two *closed halfspaces bounded* by H_a are

$$H_a^+ = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle \leq 0\}.$$

Recall that a *polyhedral cone* $Q \subset \mathbb{R}^n$ is the intersection of a finite number of closed halfspaces of the form H_a^+. If $\mathcal{A} = \{\beta_1, \ldots, \beta_q\}$ is a finite set of points in $\mathbb{R}^n$ the *cone* generated by $\mathcal{A}$, denoted by $\mathbb{R}_+\mathcal{A}$, is defined as

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i \beta_i \,\middle|\, a_i \in \mathbb{R}_+, \text{ for all } i \right\}$$

Where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. An important fact is that Q is a polyhedral cone in $\mathbb{R}^n$ if and only if there exists a finite set $\mathcal{A} \subset \mathbb{R}^n$ such that $Q = \mathbb{R}_+\mathcal{A}$, see [9, Theorem 4.1.1].

Definition 1.1 A *proper face* of a polyhedral cone Q is a subset $F \subset Q$ such that there is a *supporting hyperplane* H_a satisfying:

(a) $F = Q \cap H_a \neq \emptyset$,

(b) $Q \not\subset H_a$ and $Q \subset H_a^+$.

The *improper faces* of Q are Q itself and $\emptyset$.

Definition 1.2 A proper face F of a polyhedral cone $Q \subset \mathbb{R}^n$ is called a *facet* of Q if

$$\dim(F) = \dim(Q) - 1.$$

Definition 1.3 If a polyhedral cone Q is written as

$$Q = H_{a_1}^+ \cap \cdots \cap H_{a_r}^+$$

such that no one of the $H_{a_i}^+$ can be omitted, then we say that this is an *irreducible representation* of Q.

It follows from the theorem below that if a polyhedral cone Q is not an affine space and has the dimension of the ambient space, then there is only one irreducible representation of Q.

Theorem 1.4 *Let Q be a polyhedral cone in $\mathbb{R}^n$ with $\dim(Q) = n$ and such that $Q \neq \mathbb{R}^n$. Let*

$$Q = H_{a_1}^+ \cap \cdots \cap H_{a_r}^+ \qquad (*)$$

be a representation of Q with $H_{a_1}^+, \ldots, H_{a_r}^+$ distinct, where $a_i \in \mathbb{R}^n \setminus \{0\}$ for all i. Set $F_i = Q \cap H_{a_i}$, for each $i = 1, \ldots, r$. Then

(a) $\mathrm{ri}(Q) = \{x \in \mathbb{R}^n | \langle x, a_1\rangle > 0, \ldots, \langle x, a_r\rangle > 0\}$, *where* $\mathrm{ri}(Q)$ *is the relative interior of Q, which in this case is just the interior.*

(b) *Each facet F of Q is of the form $F = F_i$ for some i.*

(c) *Each F_i is a facet of Q if and only if $(*)$ is irreducible.*

Proof: See [1, Theorem 8.2] and [9, Theorem 3.2.1]. □

The following two results are quite useful to determine the facets of a polyhedral cone.

Proposition 1.5 *Let $\mathcal{A}$ be a finite set of points in $\mathbb{Z}^n$. If F is a nonzero face of $\mathbb{R}_+\mathcal{A}$, then $F = \mathbb{R}_+\mathcal{A}'$ for some $\mathcal{A}' \subset \mathcal{A}$.*

Proof: Let $F = \mathbb{R}_+\mathcal{A} \cap H_a$ with $\mathbb{R}_+\mathcal{A} \subset H_a^+$. Then F is equal to the cone generated by the set $\mathcal{A}' = \{\alpha \in \mathcal{A} | \langle \alpha, a\rangle = 0\}$. □

Corollary 1.6 *Let $\mathcal{A}$ be a finite set of points in $\mathbb{Z}^n$ and F a face of $\mathbb{R}_+\mathcal{A}$.*

(a) *If $\dim F = 1$ and $\mathcal{A} \subset \mathbb{N}^n$, then $F = \mathbb{R}_+\alpha$ for some $\alpha \in \mathcal{A}$.*

(b) *If $\dim \mathbb{R}_+\mathcal{A} = n$ and F is a facet defined by the supporting hyperplane H_a, then H_a is generated by a linearly independent subset of $\mathcal{A}$.*

Definition 1.7 Let Q be a polyhedral cone in $\mathbb{R}^n$ with $\dim(Q) = n$ and such that $Q \neq \mathbb{R}^n$. Let

$$Q = H_{a_1}^+ \cap \cdots \cap H_{a_r}^+ \tag{$*$}$$

be the irreducible representation of Q. If $a_i = (a_{i1}, \ldots, a_{in})$ we call

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0,\ i = 1, \ldots, r$$

the *equations of the cone* Q.

Remark 1.8 If $Q = \mathbb{R}_+\alpha_1 + \cdots + \mathbb{R}_+\alpha_q \neq \mathbb{R}^n$ with $\alpha_i \in \mathbb{Q}^n$ for all i and $\dim(Q) = n$, then it is not hard to prove that there are unique (up to sign) $a_1, \ldots, a_r$ in $\mathbb{Z}^n$ with relative prime entries and such that $(*)$ is the irreducible representation of Q. Indeed note that if H_a is a supporting hyperplane generated by a subset of $n-1$ linearly independent vectors in $\{\alpha_1, \ldots, \alpha_q\}$, then H_a has an orthogonal basis of vectors in $\mathbb{Q}^n$ and consequently there is a normal vector b to H_a such that $b \in \mathbb{Q}^n$ and $H_a = H_b$.

The main result First let us fix some of the notation that will be used throughout the remaining of this note. Let K be a field and

$$R = K[X_1, \ldots, X_n]$$

a polynomial ring with coefficients in K. Given two positive integers k, n with $k \leq n$ we define

$$F_{n,k} = \{\{i_1, i_2, \ldots, i_k\} \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\},$$

note that $|F_{n,k}| = \binom{n}{k}$.

We will use the notation $X^{\{i_1,\ldots,i_k\}}$ for the monomial $X_{i_1}X_{i_2}\cdots X_{i_k}$, where $\{i_1, \ldots, i_k\} \in F_{n,k}$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a vector with non negative integral entries we will set $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $\log(X^\alpha) = \alpha$.

Let $F = \{X^\alpha | \alpha \in F_{n,k}\}$ be the set of *square-free k-products*. The *Rees algebra* of the ideal $I = \langle F \rangle$ will be denoted by

$$\begin{aligned} \mathcal{R}(I) &= K[X_1, \ldots, X_n, FT] \\ &= K[X_1, \ldots, X_n, f_1T, \ldots, f_{\binom{n}{k}}T] \subset R[T], \end{aligned}$$

where T is a new variable and $F = \{f_1, \ldots, f_{\binom{n}{k}}\}$. We can make this algebra standard with the following graduation $\deg(X_i) = 1$ and $\deg(T) = 1 - k$, that is, with this graduation $\deg(f_iT) = 1$ for all i.

Let $\{\widehat{e}_1, \ldots, \widehat{e}_{n+1}\}$ be the canonical base of $\mathbb{R}^{n+1}$. Given $A \subset \mathbb{N}^{n+1}$ finite, we define C_A to be the subsemigroup of $\mathbb{N}^{n+1}$ generated by A:

$$C_A = \sum_{\alpha \in A} \mathbb{N}\alpha,$$

thus the *cone* generated by C_A is:

$$\mathbb{R}_+ C_A = \mathbb{R}_+ A = \left\{ \sum a_i \gamma_i \mid a_i \in \mathbb{R}_+, \gamma_i \in A \right\},$$

where $\mathbb{R}_+$ denote the set of non negative real numbers.

With this notation we state our main result:

Theorem 1.9 *Let $A = \{\log(g) \mid g \in \{X_1, \ldots, X_n, FT\}\}$. One has:*

(a) *If $n = k$ and $N = \{\widehat{e}_1 - \widehat{e}_{n+1}, \ldots, \widehat{e}_n - \widehat{e}_{n+1}, \widehat{e}_{n+1}\}$, then*

$$\mathbb{R}_+ C_A = \bigcap_{a \in N} H_a^+$$

is the irreducible representation of $\mathbb{R}_+ C_A$.

(b) *Assume $n > k$. For $\{i_1, \ldots, i_r\} \in F_{n,r}$ and $0 < r < k$ define the vectors*

$$e_{i_1 \cdot \ldots \cdot i_r} = (1, \ldots, 1, \overset{i_1}{0}, 1, \ldots, 1, \overset{i_2}{0}, 1, \ldots, 1, \overset{i_r}{0}, 1, \ldots, 1, \overset{n+1}{r - k}),$$

and define $e_\phi = (1, \ldots, 1, -k)$ if $r = 0$. If

$$N = \{\widehat{e}_1, \ldots, \widehat{e}_{n+1}, e_{i_1 \cdot \ldots \cdot i_r} \mid \{i_1, \ldots, i_r\} \in F_{n,r}, 0 \leq r < k\},$$

then

$$\mathbb{R}_+ C_A = \bigcap_{\alpha \in N} H_a^+$$

is the irreducible representation of $\mathbb{R}_+ C_A$.

Proof: Case (a): In this case

$$A = \{\widehat{e}_1, \ldots, \widehat{e}_n, \widehat{e}_1 + \widehat{e}_2 + \cdots + \widehat{e}_{n+1}\}.$$

Clearly

$$n + 1 = \operatorname{rank}(M_A) = \dim \mathbb{R}_+ C_A,$$

where M_A is the matrix generated by the elements of A as rows. We must show that $H_a \cap \mathbb{R}_+ C_A$ with $a \in N$ are precisely the facets of the cone $\mathbb{R}_+ C_A$.

Obviously $\mathbb{R}_+ C_A \subset H_a^+ \;\; \forall a \in N$, now let see that

$$\dim H_a \cap \mathbb{R}_+ C_A = n \quad \forall a \in N.$$

The case $a = \widehat{e}_{n+1}$ is easy because $\{\widehat{e}_1, \ldots, \widehat{e}_n\} \subset H_{\widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A$, and then $H_{\widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A$ is a facet.

For the other cases take $a = \widehat{e}_i - \widehat{e}_{n+1}$. Let $j \in \{1, \ldots, n\}$. Observe that

$$\begin{aligned} \langle \widehat{e}_i - \widehat{e}_{n+1}, \widehat{e}_j \rangle &= \delta_{ij} \;\; \forall j \\ \langle \widehat{e}_i - \widehat{e}_{n+1}, \widehat{e}_1 + \cdots + \widehat{e}_{n+1} \rangle &= 0, \end{aligned}$$

then $\{\widehat{e}_1, \ldots, \widehat{e}_{i-1}, \widehat{e}_{i+1}, \ldots, \widehat{e}_n, \widehat{e}_{n+1}\} \subset H_{\widehat{e}_i - \widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A$ and consequently

$$H_{\widehat{e}_i - \widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A \quad \text{is a facet} \quad \forall i = 1, \ldots n.$$

That they are all the facets follows by the fact that A has $n+1$ linearly independent elements and the facets need n of them by Corollary 1.6(b), hence there can only exist $n+1$ facets, and they were already given.

Hence, if $n = k$ then $\mathbb{R}_+ C_A = \cap_{a \in N} H_a^+$ is the irreducible representation by Theorem 1.4, as wanted.

Case (b): Let

$$f_{j_1 \cdot \ldots \cdot j_k} = (0, \ldots, 0, \overset{j_1}{1}, 0, \ldots, 0, \overset{j_2}{1}, 0, \ldots, 0, \overset{j_k}{1}, 0, \ldots, 0, \overset{n+1}{1})$$

for $\{j_1, \ldots, j_k\} \in F_{n,k}$. In this case one has

$$A = \{\widehat{e}_1, \ldots, \widehat{e}_n, f_{j_1 \cdot \ldots \cdot j_k} \mid \{j_1, \ldots, j_k\} \in F_{n,k}\}.$$

First let us see that if

$$N = \{\widehat{e}_1, \ldots, \widehat{e}_{n+1}, e_{i_1 \cdot \ldots \cdot i_r} \mid \{i_1, \ldots, i_r\} \in F_{n,r}, \;\; 0 \le r < k\},$$

then $H_a \cap \mathbb{R}_+ C_A$ is a facet $\forall a \in N$.

Observe that we have:

$$\langle e_{i_1 \cdot \ldots \cdot i_r}, f_{j_1 \cdot \ldots \cdot j_k} \rangle = \text{Card } \left(\{i_1, \ldots, i_r\} \cap (\{1, \ldots, n\} \setminus \{j_1, \ldots, j_k\}) \right),$$

where the left hand side of the equality is an inner product, then it follows easily that $\mathbb{R}_+ C_A \subset H_a^+ \;\; \forall a \in N$.

Now to see that $H_a \cap \mathbb{R}_+ C_A$ is a facet for any a in N we need only show that they have the correct dimension n.

We have $\{\widehat{e}_1, \ldots, \widehat{e}_n\} \subset H_{\widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A$, hence $H_{\widehat{e}_{n+1}} \cap \mathbb{R}_+ C_A$ is a facet. We also have $\{\widehat{e}_1, \ldots, \widehat{e}_{i-1}, \widehat{e}_{i+1}, \ldots \widehat{e}_n\} \subset H_{\widehat{e}_i} \cap \mathbb{R}_+ C_A$, hence we only need another vector linearly independent to the $n-1$ given vectors. If we choose $\{j_1, \ldots, j_k\} \in F_{n,k}$ with $i \notin \{j_1, \ldots, j_k\}$ (we can because $n > k$), then $f_{j_1 \cdot \ldots \cdot j_k} \in H_{\widehat{e}_i} \cap \mathbb{R}_+ C_A$ and hence $H_{\widehat{e}_i} \cap \mathbb{R}_+ C_A$ is a facet $\forall i = 1, \ldots, n$.

Let us study now $H_{e_{i_1 \cdot \ldots \cdot i_r}} \cap \mathbb{R}_+ C_A$ with $\{i_1, \ldots, i_r\} \in F_{n,r}$ and $0 \leq r < k$, clearly we have that

$$\{\widehat{e}_{i_1}, \ldots, \widehat{e}_{i_r}\} \subset H_{e_{i_1 \cdot \ldots \cdot i_r}} \cap \mathbb{R}_+ C_A$$

so we only need to show another $n-r$ vectors in $H_{e_{i_1 \cdot \ldots \cdot i_r}} \cap \mathbb{R}_+ C_A$ which are linearly independent to $\{\widehat{e}_{i_1}, \ldots, \widehat{e}_{i_r}\}$. To choose those $n-r$ linearly independent vectors first observe that

$$\dim \mathcal{L}\{f_{j_1 \cdot \ldots \cdot j_k} \mid \{j_1, \ldots, j_k\} \in F_{n,k}\} = n$$

but we have that:

$$\langle e_{i_1 \cdot \ldots \cdot i_r}, f_{j_1 \cdot \ldots \cdot j_k} \rangle = 0 \Longleftrightarrow \{i_1, \ldots, i_r\} \subset \{j_1, \ldots, j_k\}$$

Observe that given a set $\{i_1, \ldots, i_r\} \in F_{n,r}$ we can choose $\{j_1, \ldots, j_k\}$ in $F_{n,k}$ satisfying $\{i_1, \ldots, i_r\} \subset \{j_1, \ldots, j_k\}$ in $\binom{n-r}{k-r}$ distinct forms. Consider the vector space

$$W = \mathcal{L}(\{f_{j_1 \cdot \ldots \cdot j_k} \mid \{j_1, \ldots, j_k\} \in F_{n,k} \text{ and } \{i_1, \ldots, i_r\} \subset \{j_1, \ldots, j_k\}.\}$$

Since $\dim_{\mathbb{R}}(W) \geq n - r$ and $\binom{n-r}{k-r} \geq n - r$ we can choose the $n - r$ linearly independent vectors needed, hence

$$H_{e_{i_1} \cdot \ldots \cdot e_{i_r}} \cap \mathbb{R}_+ C_A$$

is a facet for all $\{i_1, \ldots, i_r\} \in F_{n,r}$ with $0 \leq r < k$.

Finally let us see that the

$$(n+1) + \sum_{r=0}^{k-1} \binom{n}{r}$$

facets given are all the facets of the cone $\mathbb{R}_+ C_A$.

Let F be a facet of the cone $\mathbb{R}_+C_A$, hence there exist $\alpha_1,\ldots,\alpha_n \in A$ linearly independent vectors and $0 \neq b \in \mathbb{R}^{n+1}$ such that $F = \mathbb{R}_+C_A \cap H_b$, $\mathcal{L}\{\alpha_1,\ldots,\alpha_n\} = H_b$, and $\mathbb{R}_+C_A \subset H_b^+$.

Let see that exist $a \in N$ such that $\mathcal{L}\{\alpha_1,\ldots,\alpha_n\} = H_a$ and $\mathbb{R}_+C_A \subset H_a^+$.

We see this in three cases: Let $B = \{\alpha_1,\ldots,\alpha_n\}$

1. If $B = \{\widehat{e}_1,\ldots,\widehat{e}_n\}$ we can take $a = \widehat{e}_{n+1}$.
2. If $B \subset \{f_{j_1\cdot\ldots\cdot j_k} \mid \{j_1,\ldots,j_k\} \in F_{n,k}\}$, it is enough to take $a = e_\phi$.
3. If $B = \{\widehat{e}_{i_1},\ldots,\widehat{e}_{i_s}, f_{j_1^1\cdot\ldots\cdot j_k^1},\ldots,f_{j_1^t\cdot\ldots\cdot j_k^t}\}$ with $s,t>0$ $(s+t=n)$.

Here $1 \leq i_1 < \cdots < i_s \leq n$ and $\{j_1^1,\ldots,j_k^1\},\ldots,\{j_1^t,\ldots,j_k^t\} \in F_{n,k}$. In this final case we will show that $\{i_1,\ldots,i_s\} \subset \{j_1^m,\ldots,j_k^m\}$ $\forall m$.

By contradiction, suppose that exist $\widehat{e}_{i_p} \in B$ and $f_{j_1^q\cdot\ldots\cdot j_k^q} \in B$ with $i_p \notin \{j_1^q,\ldots,j_k^q\}$ note that there exists f_{β_i}, with $\beta_i \in F_{n,k}$, such that:

$$\widehat{e}_{i_p} + f_{j_1^q\cdot\ldots\cdot j_k^q} = \widehat{e}_{j_i^q} + f_{\beta_i} \quad \forall i = 1,\ldots,k$$

Observe that $\langle \widehat{e}_{i_p} + f_{j_1^q\cdot\ldots\cdot j_k^q}, b\rangle = 0$, then

$$\langle \widehat{e}_{j_i^q}, b\rangle = -\langle f_{\beta_i}, b\rangle.$$

Hence $\langle \widehat{e}_{j_i^q}, b\rangle = 0$ because in the other case $\widehat{e}_{j_i^q}$ and f_{β_i} would be in opposite sides of H_b and that can not be. Therefore $\widehat{e}_{j_i^q} \in H_b$ $\forall i = 1,\ldots,k$, consequently $\langle f_{j_1^q\cdot\ldots\cdot j_k^q}, b\rangle = b_{n+1} = 0$.

Now as M_B has rank n (M_B is the matrix whose rows are the elements of B), then via row reduction M_B takes the form $[I_n, C]$, where C is an $n \times 1$ matrix. We have already proven that $b_{n+1} = 0$ and the reduction shows that $b_1 = \cdots = b_n = 0$, this is a contradiction because $b \neq 0$. Then

$$\{i_1,\ldots,i_s\} \subset \{j_1^m,\ldots,j_k^m\} \ \ \forall m = 1,\ldots,t$$

from this we conclude that it is enough to take $a = e_{i_1\cdot\ldots\cdot i_s}$ then the facets given are all the facets of the cone and the representation is irreducible. □

Remark 1.10 Note that in Proposition 1.9 (b) the number of vectors in N is equal to

$$n + 1 + \sum_{r=0}^{k-1}\binom{n}{r}.$$

2 Computing the type and the a-invariant

At the end we illustrate with an example how to compute the a-invariant and the type of $\mathcal{R}(I)$ using the equations of the cone. For use below we recall that $\mathcal{R}(I)$ is a normal domain according to [8].

Definition 2.1 Let R be a polynomial ring over a field k and F a finite set of monomials in R. A decomposition

$$k[F]=\bigoplus_{i=0}^{\infty}k[F]_i$$

of the k-vector space $k[F]$ is an *admissible grading* if $k[F]$ is a positively graded k-algebra with respect to this decomposition and each component $k[F]_i$ has a finite k-basis consisting of monomials.

Theorem 2.2 (Danilov,Stanley) *Let $R=k[x_1,\ldots,x_n]$ be a polynomial ring over a field k and F a finite set of monomials in R. If $k[F]$ is normal, then the canonical module $\omega_{k[F]}$ of $k[F]$, with respect to an arbitrary admissible grading, can be expressed as*

$$\omega_{k[F]}=(\{x^a\,|\,a\in\mathbb{N}\mathcal{A}\cap\mathrm{ri}(\mathbb{R}_+\mathcal{A})\}), \tag{1}$$

where $\mathcal{A}=\log(F)$ and $\mathrm{ri}(\mathbb{R}_+\mathcal{A})$ denotes the relative interior of $\mathbb{R}_+\mathcal{A}$.

The formula above represents the canonical module of $k[F]$ as an ideal of $k[F]$ generated by monomials. For a comprehensive treatment of the Danilov-Stanley formula see [3, Theorem 6.3.5].

Let K be a field and S a Cohen-Macaulay standard K-algebra. One can represent S as $S=R/I$, where R is a polynomial ring with the usual grading and I is a graded ideal. Recall that the type of S is the minimum number of generators of the canonical module ω_S of S, which is also equal to the last Betti number in the minimal resolution of S as an R-module. We also recall that the a-invariant of S is the degree (as a rational function) of the Hilbert series of S, which is also equal to

$$a(S)=-\min\{i\,|\,(\omega_S)_i\neq 0\}.$$

Thus it is clear that from the canonical module of S one can extract important information about the resolution of $S=R/I$ and about the Hilbert series and the Hilbert function of S.

Example 2.3 If $n = 5$ and $k = 3$, then we have 22 equations defining the cone $\mathbb{R}_+ C_A$:

$$\begin{array}{ll}
X_1 + X_2 + X_3 + X_4 + X_5 - 3T \geq 0 & T \geq 0 \\
X_1 + X_2 + X_3 + X_4 - 2T \geq 0 & X_1 \geq 0 \\
X_1 + X_2 + X_3 + X_5 - 2T \geq 0 & X_2 \geq 0 \\
X_1 + X_2 + X_4 + X_5 - 2T \geq 0 & X_3 \geq 0 \\
X_1 + X_3 + X_4 + X_5 - 2T \geq 0 & X_4 \geq 0 \\
X_2 + X_3 + X_4 + X_5 - 2T \geq 0 & X_5 \geq 0 \\
X_1 + X_2 + X_3 - T \geq 0 & \\
X_1 + X_2 + X_4 - T \geq 0 & \\
X_1 + X_2 + X_5 - T \geq 0 & \\
X_1 + X_3 + X_4 - T \geq 0 & \\
X_1 + X_3 + X_5 - T \geq 0 & \\
X_1 + X_4 + X_5 - T \geq 0 & \\
X_2 + X_3 + X_4 - T \geq 0 & \\
X_2 + X_3 + X_5 - T \geq 0 & \\
X_2 + X_4 + X_5 - T \geq 0 & \\
X_3 + X_4 + X_5 - T \geq 0 &
\end{array}$$

Note that the canonical module of $\mathcal{R}(I)$ is minimally generated by

$$\{x_1x_2(x_3x_4x_5T)\} \cup \{x_ix_jx_1x_2x_3x_4x_5T^2 | \, 1 \leq i < j \leq 5\}.$$

This assertion can be readily verified by applying the Danilov-Stanley formula and using that the relative interior of $\mathbb{R}_+ C_A$ (which in our case is the usual interior) is computed replacing $\geq$ by $>$ in the above set of inequalities (see Theorem 1.4). Since all those monomials have degree 3, the a-invariant of $\mathcal{R}(I)$ is -3 and type of $\mathcal{R}(I)$ is equal to 11.

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