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Geometry and dynamics of the residue theorem *

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Abstract

Using singular flat metrics associated to meromorphic differential forms on Riemann surfaces, a converse of the classical residue theorem is due. Also a dynamical interpretation of residues using real geodesic flows is given.

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1 Complex analysis by cut and paste methods

For the construction of nice manifolds in differential geometry, the cut and paste method is useful and very easy. This can be summarized as follows:

i) Choose some nice geometric material, for example a space form.

ii) Cut simple pieces from the material, this usually means with geodesic boundary.

iii) Define suitable paste methods for joining pieces together in a C^∞ fashion.

iv) Construct all the possible geometric objects using (i)–(iii).

The challenge is to describe and classify how many manifolds can appear. For this the description of *hidden laws* arising from (i)–(iii) is

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very useful, by this we understand some geometric laws that all the resulting objects satisfy and such that they are implicit in (i)–(iii). Recall as examples of hidden laws the Gauss–Bonnet Theorem for compact surfaces with a Riemannian metric and the Mostow's Rigidity Theorem for n–dimensional hyperbolic manifolds, $n \geq 3$.

We sketch here the above program for the study of meromorphic differential forms and vector fields as differential geometric objects on Riemann surfaces. Our goal is the description of the Residue Theorem as a hidden law.

The starting point is that over $\mathbb{C}$ or in more generality over a Riemann surface, there exists a natural one to one correspondence between meromorphic differential forms and meromorphic vector fields. Namely, given a vector field X, there is a uniquely define differential form ω requiring $\omega(X) \equiv 1$. In local coordinates this is

$$X = f(z)\frac{\partial}{\partial z} \quad \longleftrightarrow \quad \omega = \frac{dz}{f(z)} \ .$$

Where by $f(z)\frac{\partial}{\partial z}$ we have in mind $f(z)$ as a complex tangent vector to M at z, see [7] p. 340 for the formal definition.

Let (M, J) be a Riemann surface (here $J : TM \to TM$, $J^2 = -Id$ yields the complex structure) provided with a meromorphic vector field X. Then M has a canonical geometric structure defined over $M - \{zeros\ and\ poles\ of\ X\}$ given by the C^∞ flat Riemannian metric

$$g = \begin{pmatrix} \frac{1}{u^2+v^2} & 0 \\ 0 & \frac{1}{u^2+v^2} \end{pmatrix} , \qquad \text{where as usual} \ \ X = (u + \sqrt{-1}v)\frac{\partial}{\partial z} \ .$$

The recognition of these implicit *flat polyhedral structures* allows for application of the cut and paste method. Roughly speaking, we have a correspondence between the following:

$$\begin{array}{ccc} Pairs; & & Flat\ \ polyhedral \\ a\ \ Riemann\ \ surface\ \ (M, J)\ \ and & \longleftrightarrow & structures\ \ on \\ a\ \ meromorphic\ \ vector\ \ field\ \ X & & M - \{discrete\ \ set\} \end{array}$$

where the discrete set on the right comes from the zeros and poles of X.

We address the following questions:

What kind of flat polyhedral structures can appear?

What hidden laws do they obey?

Another tool is the dynamics of the meromorphic vector field X. Define its associated real vector field as $\Re e(X) = X + (\overline{X})$, considering $f(z)$ as a real tangent vector field to M at z. The classical theory of (real) differential equations allows us to construct a dynamical system $\phi : \mathbb{R} \times M \to M$, coming from the flow of the vector field (strictly speaking ϕ can be defined only almost everywhere in $\mathbb{R} \times M$ for generic X).

The blending of the polyhedral structure g and the vector field $\Re(X)$ has very interesting properties.

For example, the trajectory solutions of $\Re e(X)$ are geodesics of g. Another key point is that the zeros and poles of the meromorphic vector field give origin to singularities in the flat polyhedral structures, introducing a lot of flexibility and richness. In particular, from the dynamical point of view, simple zeros of a meromorphic vector field give origin to sources, centers or sinks of $\Re e(X)$, see Section 3.

The classical Residue Theorem asserts that the sum of the residues of any meromorphic differential form ω on a compact Riemann surface is zero. Note that for simple singularities, information about the residues of ω changes to information on the linear parts at the zeros of X. Some geometrical and dynamical flavor is described by:

Theorem A. *Let X be a meromorphic vector field on a compact Riemann surface (M, J), having only simple zeros with linear parts $a_1, \dots, a_s \in \mathbb{C}$. Consider $\Re e(X) = X + \overline{X}$ its real associated vector field, having ϕ_1 as time–1 flow, and g the singular flat Riemannian metric on $M - \{zeros\ and\ poles\ of\ X\}$ as above. Then*
1.- ϕ_1 leaves invariant g.
2.- The signed g–area coming from (and respectively falling into) a source p_i (respectively a sink) under ϕ_1 is

$$2\pi \Re e \left(\frac{1}{a_j} \right) ,$$

where $a_j^{-1} = \lambda_j$ is the residue at p_i of the associated differential form.
3.- The map ϕ_1 on M satisfies the g–area Conservation Law,

$$\sum_{p_i \text{ is a source}} [g\text{−area coming from } p_i] + \sum_{p_j \text{ is a sink}} [g\text{−area falling in } p_j] = 0 .$$

Figure 1: The Residue Theorem as a Conservation Law.

The g–area coming from a source p_i under ϕ_1 is defined as follows. Consider γ a suitable small simple loop, enclosing p_i and no other singularities of $\Re e(X)$, and assume that γ is transverse to $\Re e(X)$. Then, both γ and $\phi_1(\gamma)$ form the boundary of an annulus, its g–area is the desired number, see Figure 1. Note that for a source the area is positive, and for a sink this is negative. In Section 4 we give the equivalence between the Conservation Law and the classical Residue Theorem.

The converse of Theorem A is as follows:

Theorem B. *Let M be a compact orientable C^∞ two–manifold, given $a_1, ..., a_s$ in $\mathbb{C}$, $s \geq 2$, numbers such that $\sum a_j^{-1} = 0$. Then there exists a complex structure J and a meromorphic vector field X on M having simple zeros in s different points $\{p_1, ..., p_s\} \subset M$ and linear parts $a_j = X'(p_j)$.*

The proofs given here only involve the geometry and dynamics of X, described above. For a classical proof of Theorem B, see [3] p. 52.

The above flat polyhedral structures appear as a main subject in quadratic differential theory, a highly developed and useful area in complex analysis, see [1], [5], [12]. Our point of view is that elementary polyhedral structures coming from holomorphic functions can help to explain some basic facts in complex analysis. In order to reduce the necessary background of the paper we start working here with meromorphic differential forms.

2 Complex integration, flat metrics and American football

Following differential geometric ideas, we describe the local model for a domain $\Omega \subset \mathbb{C}$ provided with a meromorphic differential form ω (at regular points). Assume without loss of generality, that ω is written as dz/f, where $f = u + \sqrt{-1}v$, and $p_0 \in \Omega$ is a regular point. Let $V \subset \Omega$ be an open disk around p_0 free of zeros and poles and consider the holomorphic map:

$$h(p) = \int_{p_0}^{p} \frac{dz}{f(z)} : V \subset \Omega \to \mathbb{C} \ .$$

A natural question is for which real trajectories starting at p_0,
is the value of the integral purely real
or similarly,
is the value of the integral purely imaginary?

Consider the associated meromorphic vector field $X = (u+\sqrt{-1}v)\frac{\partial}{\partial z}$, and define its real and imaginary parts as the real vector fields:

$$\Re e(X) \doteq (X + \overline{X}) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

$$\Im m(X) \doteq (\sqrt{-1}X - \sqrt{-1}\overline{X}) = -v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y} \ ,$$

here $\overline{X}$ means the conjugate. They describe the real and imaginary times of the dynamical system defined by X. In others words assume $\alpha(t), \beta(s) : (-\epsilon, \epsilon) \subset \mathbb{R} \to \Omega$ are local non–singular trajectories of $\Re e(X)$ and $\Im m(X)$ respectively, with $\alpha(0) = \beta(0) = p_0$, then

$$\int_{p_0}^{\alpha(t)} \frac{dz}{f(z)} = t \in \mathbb{R} \ ,$$

$$\int_{p_0}^{\beta(s)} \frac{dz}{f(z)} = \sqrt{-1}s \in \sqrt{-1}\mathbb{R} \ .$$

Now we compute the bracket of $\Re e(X)$ and $\Im m(X)$.
Starting at p_0: follow the trajectory of $\Re e(X)$ through p_0 for time t.
Then, follow the trajectory of $\Im m(X)$ for time s, then the trajectory of $\Re e(X)$ backwards for time $-t$.
Finally, follow the trajectory of $\Im m(X)$ backwards for time $-s$.
Have we retuned to the original point p_0?
If γ is the above trajectory, then

$$\int_{\gamma} \frac{dz}{f(z)} = t + \sqrt{-1}s - t - \sqrt{-1}s = 0 \ .$$

Since $1/f(p_0) \in \mathbb{C} - \{0\}$, the inverse function theorem says that $h = \int(dz/f)$ is locally an invertible map, and we are in the original point p_0. The vector fields $\Re e(X)$ and $\Im m(X)$ commute.

They are an orthonormal frame for the Riemannian metric g (as in Section 1) in Ω, this is

$$g(\Re e(X), \Re e(X)) = g(\Im m(X), \Im m(X)) = 1 \ , \quad g(\Re e(X), \Im m(X)) = 0 \ .$$

It is a very well known result that a metric having an orthonormal frame that commutes is flat, [12] p. 261, hence the curvature of g is zero. Let us give an example.

Pictorial description of the flat metric. Remember an American football game. The field is an Euclidean rectangle in $\mathbb{R}^2$ furnished with a family of parallel lines $\{x = constant\}$. On each play, the players try to carry the ball across these lines. Only the horizontal advance is significant, independent of the trajectory γ that describes the player with the ball. In particular, the advance for a closed trajectory is zero. Using path integral, a mathematical synthesis is

$$\begin{array}{ccc} one \;\; play & \to & horizontal \;\; advance \\ \gamma & \mapsto & \int_\gamma dx \end{array} .$$

The case of complex integrals generalize the game. Locally we have a rectangle furnished with horizontal and vertical lines (the trajectories of $\Re e(X)$, and $\Im m(X)$). On each play the players try to carry the ball across these lines. In addition, now horizontal and vertical advances are significant. They are independent of the trajectory, depending only on the start and end points. In symbols

$$\begin{array}{ccc} one \;\; play & \to & horizontal \;\; and \;\; vertical \;\; advances \\ \gamma & \mapsto & \left(\Re e\left(\int_\gamma \frac{dz}{f(z)}\right), \Im m\left(\int_\gamma \frac{dz}{f(z)}\right)\right) \end{array} .$$

We leave for the imaginative reader the description of field goals and scores.

If γ is in a rectangle having as sides trajectories of $\Re e(X)$ and $\Im m(X)$ and has end points p and q, then

$$d(p, q) = \sqrt{\left(\Re e\left(\int_\gamma \frac{dz}{f(z)}\right)\right)^2 + \left(\Im m\left(\int_\gamma \frac{dz}{f(z)}\right)\right)^2}$$

Figure 2: Complex integration and american football.

is an Euclidean distance on the rectangle.

In mathematical language the above is explained in the following result.

Corollary. *Let $\Omega \subset \mathbb{C}$ be a domain, under a local holomorphic change of coordinates:*
1.- If $p_0 \in \Omega$ is a regular value of a meromorphic differential form $\omega = dz/f(z)$ on Ω, then ω is equivalent with dz.
2.- If $p_0 \in \Omega$ is a regular value of a meromorphic vector field $X = f(z)\frac{\partial}{\partial z}$ on Ω, then X is equivalent with $\frac{\partial}{\partial z}$.
In both cases the holomorphic change of coordinates is given by $h(p) = \int_{p_0}^{p}(dz/f)$.

Proof: For (2) obviously,

$$\left(\int_{p_0}^{p} \frac{dz}{f(z)}\right)_* f(z)\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \ ,$$

in the language of differential equations, $h(p)$ is a flow box. □

The differential geometric meaning can be found below.

Corollary. *Let $\Omega \subset \mathbb{C}$ be a domain, with a meromorphic differential form $dz/f(z)$, or equivalently with a meromorphic vector field $f(z)\frac{\partial}{\partial z}$, then there exists in $\Omega - \{zeros\ and\ poles\ of\ f\}$ a C^∞ flat Riemannian* *metric g, such that the maps*

$$\begin{array}{rcl} \int \frac{dz}{f} : V \subset (\Omega, g) & \to & (\mathbb{C}, \delta) \\ p & \mapsto & \int_{p_0}^{p} \frac{dz}{f(z)} \end{array}$$

are local isometries, where V is a disk around p_0 free of zeros and poles, and δ is the usual flat metric in $\mathbb{C}$.
Moreover, the trajectories of $\Re e(X)$ and $\Im m(X)$ are geodesics of g.

We can now introduce the *cut and paste method*:

i) Choose as material the Euclidean plane $(\mathbb{C}, \delta)$.

ii) Cut rectangles from the material, having (usual) horizontal and vertical trajectories as boundary.

iii) Define the paste procedure as follows; gluing rectangles to get topological two–manifolds and so that the sides in the boundary of the rectangles are glued by isometries in such a way that horizontal and vertical trajectories are well–defined over the boundary of the rectangles.

This must produce pairs (Ω, ω), where Ω is a Riemann surface and ω is a meromorphic differential form.

The simplest examples appear below

Example. Consider $\omega = dz/z$ on $\Omega = \mathbb{C}$. The resulting flat metric on $\mathbb{C} - \{0\}$ is isometric to the Euclidean cylinder $S^1_{2\pi} \times \mathbb{R}$, where 2π means the g–length of the closed geodesics; they correspond to trajectories of $\Im m(z\frac{\partial}{\partial z})$. See the zero of order one in Figure 3.

Example. Consider $\omega = zdz$ on $\Omega = \mathbb{C}$. The resulting flat metric on $\mathbb{C} - \{0\}$ is isometric to the gluing of four copies of flat half planes along the boundaries; the metric has as geodesics the trajectories of $\Re e(\frac{1}{z}\frac{\partial}{\partial z})$. See the pole of order one in Figure 3.

Further examples with computer plots appear in [2], [9] and [10].

The next step is the description at the singular points.

3 Zeros and poles of meromorphic vector fields

Start with the local normal forms.

Lemma. *Let X be a meromorphic vector field on a neighborhood of $0 \in \mathbb{C}$, with respect to a local holomorphic change of coordinates:*
1.- If 0 is a pole of order k for X, then X is equivalent with

$$z^{-k}\frac{\partial}{\partial z} \ .$$

2.- If 0 *is a simple zero for* X*, then* X *is equivalent with*

$$az\frac{\partial}{\partial z}\ ,$$

where $a = X'(0)$ *is the linear part.*
3.- If 0 *is a zero of order* $k \geq 2$ *for* X*, then* X *is equivalent with*

$$\frac{z^k}{1+\lambda z^{k-1}}\frac{\partial}{\partial z}\ ,$$

where λ *is the residue of the associated meromorphic differential form at* 0*.*

Proof: As usual, the reduction of X to the normal form is realized by means of a suitable holomorphic change of coordinates in power series. See [2] or [12] p. 27–37 for the explicit computations.

Moreover, the correspondences between meromorphic; vector fields, differential forms and quadratic differentials

$$f(z)\frac{\partial}{\partial z} \ \leftrightarrow\ \frac{dz}{f(z)} \ \rightarrow\ \frac{dz^2}{f(z)^2}\ ,$$

are well defined in Riemann surfaces. Hence the vector field $f(z)\frac{\partial}{\partial z}$ can be translated into the meromorphic quadratic differential $dz^2/f(z)^2$, in order to apply the theory in [12]. □

Recall that in cases (2) and (3) the associated differential forms are

$$\frac{dz}{az} \doteq \frac{\lambda dz}{z}\ ,\ \text{and}\ \left(\frac{1}{z^k}+\frac{\lambda}{z}\right)dz\ .$$

The geometry and dynamics of the singularities are as follows.

Consider in the sphere $S^2 = \mathbb{C}\cup\{\infty\}$ the closure of the half plane $\overline{H}^2 = \{x+\sqrt{-1}y \mid y \geq 0\} \subset S^2$, we consider $\overline{H}^2$ with its natural flat metric and foliated by trajectories of $\frac{\partial}{\partial x}$. We define a *flat elliptic sector* as the intersection of an open neighborhood of $\infty \in S^2$ and $\overline{H^2}$. Similarly a *flat hyperbolic sector* is the intersection of an open neighborhood of $0 \in S^2$ and $\overline{H^2}$. By definition it has cone angle π.

Corollary. *Let* X *be a meromorphic vector field on a neighborhood of* $0 \in \mathbb{C}$*.*

1.- If 0 *is a pole of order* $-k$ *for* X*.*
Then, the singular flat metric associated to X *has cone angle* $(2k+2)\pi$ *at* 0*.*

The trajectories of $\Re e(X)$ *define* $2k+2$ *flat hyperbolic sectors at* 0.

2.- If 0 *is a zero of order one for* X.
Then, the singular flat metric associated to X *has a cylindrical end at* 0, *i.e. it is isometric to a cylinder* $S^1_T \times (0, \infty)$, *where* $T = (2\pi/|X'(0)|)$.
The trajectories of $\Re e(X)$ *define a source, a center or a sink at* 0, *according to whether* $\Re e(X'(0))$ *is* $>$, $=$ *or* < 0, *respectively).*

3.- If 0 *is a zero of order* $k \geq 2$ *for* X.
Then, the real trajectories of X *define* $2k-2$ *elliptic sectors at* 0. *The singular flat metric associated to* X *depends on* λ *as a continuous parameter of isometry type. In the case* $\lambda = 0$ *the metric defines a neighborhood isometric to* $2k-2$ *copies of flat elliptic sectors, glued along the* x*–axis to obtain a topological disk. The case* $\lambda \neq 0$ *is described in the proof.*
The trajectories of $\Re e(X)$ *define* $2k-2$ *flat elliptic sectors at* 0.

Figure 3: Zeros and poles of meromorphic vector fields.

Proof: The cases (1) and (2) are well–known, see [12], and Figure 3.

For example in case (2), using the above Lemma it follows that the vector field is $X = \lambda z \frac{\partial}{\partial z}$. We consider two subcases.

If $\Re e(\lambda) = 0$, then the linear part of $\Re e(X)$ has pure imaginary eigenvalues. Its trajectories are circles, i.e. closed geodesics in the metric g_X, giving origin to a center. The flow of the orthonormal vector field $\Im m(X)$ sends closed geodesics to closed geodesics. Moving a fixed closed geodesic with the flow of $\Im m(X)$ in the direction of the zero of X, we get the description of a cylinder.

If $\Re e(\lambda) \neq 0$, then we consider the rotated vector field $e^{\sqrt{-1}\theta}X$, where $\theta \in [0, 2\pi)$, such that $\Re e(e^{\sqrt{-1}\theta}\lambda) = 0$ as in the above subcase. The Riemannian metrics coming from X and $e^{\sqrt{-1}\theta}X$ are isometric. Finally note that the trajectories of $\Re e(X)$ correspond to open geodesics in the cylinder, describing a source or a sink.

Let us describe the geometry of (3) in more detail. These vector fields look like $[z^k/(1+\lambda z^{k-1})]\frac{\partial}{\partial z}$, where $k \geq 2$. We have two cases.

In the first case $\lambda = 0$.

Here, the flat metric comes from the gluing of $2k-2$ copies of flat elliptic sectors. The canonical example where $k = 2$, is $z^2\frac{\partial}{\partial z}$ in a neighborhood of $0 \in \mathbb{C}$. Under suitable stereographic projection, this is the usual flat metric in a copy of $(\mathbb{C}, \delta)$ at a neighborhood of the point at infinity. Moreover, the real trajectories of the vector field are the usual horizontal trajectories, having α and ω–limits in ∞, that corresponds to $0 \in \mathbb{C}$ under the map $z \mapsto 1/z$. They define two elliptic sectors at $0 \in \mathbb{C}$. The complex time required for a trajectory that winds around 0 in a counterclockwise direction is exactly zero.

In the second case $\lambda \neq 0$.

We explain the case of order $k = 2$, the others follow in the same way. Consider the singular flat metric from $z^2\frac{\partial}{\partial z}$ as above. Under the map $z \mapsto 1/z$, we are working in a neighborhood of the point at infinite of $\mathbb{C}$, that is a neighborhood of 0 in the original z–plane. Consider $2\pi\sqrt{-1}\lambda = a + \sqrt{-1}b \in \mathbb{C}$ as a vector based at the origin. Remove from $\mathbb{C}$ the open sets

$$\{x + \sqrt{-1}y \mid 0 < x < a\ ,\ y > 0\} \quad , \quad \{x + \sqrt{-1}y \mid 0 < y < b\ ,\ x > 0\}\ .$$

Glue the points $\sqrt{-1}y$ with $a + \sqrt{-1}y$ for $y > b$, and x with $x + \sqrt{-1}b$ for $x > a$. The resulting flat surface is topologically like an annulus. Denote it by $(\mathbb{C}^0, g_\lambda)$, where the metric comes from the usual δ. The complex time required for a trajectory that winds around the hole in $(\mathbb{C}^0, g_\lambda)$ in a counterclockwise direction is $2\pi\sqrt{-1}\lambda$, and the residue is exactly λ. The flat metric of the vector field $[z^2/(1+\lambda z)]\frac{\partial}{\partial z}$ around $z = 0$ is isometric to a neighborhood of the end of ∞ in the above $(\mathbb{C}^0, g_\lambda)$.

To produce examples in case $k \geq 3$ it is necessary to glue to the above metric g_λ suitable copies of usual elliptic sectors. We leave the details to the interested reader. □

As far as we know the case of essential singularities is almost unexplored, see [4].

The following result is the key in the geometrical construction of meromorphic fields by suitable flat structures.

Proposition. *There exists a correspondence between the following:*
1.- Pairs that include; a Riemann surface (M, J) *and a meromorphic vector field* X *having as singularities zeros and poles.*
2.- Orientable paracompact C^∞ *two–manifolds* M*, with a discrete set* $S \subset M$*, equipped with a* C^∞ *flat Riemannian metric* g *in* $M - S$*, such that*
i) g *assumes some of the above local models at the points in* S*.*
ii) The holonomy of g *is the identity (i.e. the parallel transport along all closed trajectories in* $M - S$ *is the identity map).*
iii) $M - S$ *is provided with a non–singular unitary geodesic field.*

Proof: Let us only remark on some key points.

For (1) $\Rightarrow$ (2), given X, define S as the zeros and poles of X. Since $\Re e(X)$ and $\Im m(X)$ are a parallel and orthonormal frame, it follows that the holonomy of g is trivial. Moreover, use $\Re e(X)$ to define (iii) of part (2).

For (2) $\Rightarrow$ (1), the geometry of g at the local singular model define conformal punctures. Hence the complex structure J associated to g extends over S. $\square$

For more details on the proof of the correspondence and explicit examples, see [6], [8], [9], [12], and Section 5.

Also note that the cut and paste method defined in Section 2 always produces examples as in (2) above.

4 Proof of Theorem A

Consider in $M - \{zeros\ and\ poles\ of\ X\}$ the flat metric g, defined in Section 1.

Proof of (1)

We need to show that the time–1 flow ϕ_1 of F leaves invariant g. However, this is elementary, since ϕ_1 is an Euclidean translation with respect to the flat metric g.

Proof of (2)

The computation is local. Let $X = a\frac{\partial}{\partial z}$ be a holomorphic vector field, where $a \neq 0$. Recall that the rotated vector field

$$e^{\sqrt{-1}\theta} X$$

by an angle $\theta \in [0, 2\pi)$ defines the same flat metric g that the original from X, changing by θ the slope of the vector field $\Re e(X)$, see [9]. For exactly two values of θ, say θ_0 and $\theta_0+\pi$, the trajectories of $\Re e(e^{\sqrt{-1}\theta} X)$ are closed near the singular point 0 and hence the rotated vector field produces closed geodesics in the metric cylinder defined by g around 0. The period of the closed geodesics is $2\pi/|a|$. Hence, the g–area of the ϕ_1 flow crossing by any of these trajectories is exactly

$$\frac{2\pi}{|a|}\cos(arg(a)) = 2\pi \Re e\left(\frac{1}{a}\right) \ .$$

Proof of (3)

We use some very simple dynamical ideas. Note that the time–1 flow ϕ_1 is well–defined outside of the separatrix trajectories of the poles of X (see Figure 3). The union of these separatrices is of g–area zero. Hence $\phi_1 : \mathbb{Z} \times M \to M$ defines a dynamical system almost everywhere in $\mathbb{Z} \times M$.

By removing from (M, g) some small disks around the zeros of X, the resulting two–manifold $N \subset M$ has finite g–area (see Figure 1).

Finally, note that ϕ_1 preserves the g–area in (M, g). Since the area of N is finite, the area coming in N under ϕ_1, is necessarily equal to the area leaving N under ϕ_1. The result follows. $\square$

Remark. *Equivalence between the Conservation Law and the Residue Theorem.*

Writing the Conservation Law for the family of rotated vector fields $\{e^{\sqrt{-1}\theta} X\}$, we get

$$2\pi \sum_{j=1}^{s} \Re e\left(\frac{e^{\sqrt{-1}\theta}}{a_j}\right) = 0 \ .$$

Since the above holds for every $\theta \in [0, 2\pi)$, the Residue Theorem $\sum a_j^{-1} = 0$ follows. $\square$

5 Proof of Theorem B

Given the linear parts $a_1, ..., a_s$, the idea is to find a suitable flat metric realizing the above data. The construction is by the cut and paste method.

Case 1. M is the sphere and $s = 2$.

The linear parts satisfy $a_1^{-1} + a_2^{-1} = 0$. Note that $a_1^{-1} \neq 0$. Consider the flat cylinder

$$\mathbb{C}/\{n(2\pi\sqrt{-1}a_1^{-1}) \mid n \in \mathbb{Z}\} \ .$$

Introduce the vector field $\frac{\partial}{\partial z}$; this gives the desired object. Note that for the cylindrical end p

$$residue\ (dz, p) \doteq \frac{1}{2\pi\sqrt{-1}} \int_0^{2\pi\sqrt{-1}a_1^{-1}} dz = a_1^{-1} \ .$$

The metrics satisfy $g = \delta$.

Case 2. M is the sphere and $s \geq 3$.

Order the linear parts such that

$$arg(a_1^{-1}) \leq arg(a_2^{-1}) \leq ... \leq arg(a_s^{-1}) \ .$$

Consider in $\mathbb{C}$ the unique convex polygon having s vertex at the points

$$a_0 = 0,\ 2\pi\sqrt{-1}a_1^{-1},\ 2\pi\sqrt{-1}(a_1^{-1}+a_2^{-1}), ...,\ 2\pi\sqrt{-1}(a_1^{-1}+...+a_s^{-1}) = 0 \ ,$$

equipped with the usual flat metric δ. The case where the polygon degenera–tes to some straight segment can appear. Assume for a moment that all the sides of the polygon are different among them. Identify all the vertices of the polygon to the same point, called q, obtaining a two–manifold with boundary homeomorphic to a sphere minus s open disks. Glue to each boundary component a flat cylinder isometric to $S^1_{|2\pi a_j^{-1}|} \times [0, \infty)$, using isometries, where the sub–index $|2\pi a_j^{-1}|$ means the δ–length of the closed geodesics. See Figure 4 for the case $s = 5$. The resulting flat surface has the following features: s cylindrical ends, say $\{p_j\}$, and one point q having cone angle $(2s - 2)\pi$.

The vector field $\frac{\partial}{\partial z}$ on $\mathbb{C}$ induces the desired vector field on the resulting flat surface.

In fact, for the associated meromorphic differential form note that the integral over the j–th side of the polygon satisfies

$$residue\ (dz, p_j) \doteq \frac{1}{2\pi\sqrt{-1}} \int_{2\pi\sqrt{-1}(a_1^{-1}+...+a_{j-1}^{-1})}^{2\pi\sqrt{-1}(a_1^{-1}+...+a_j^{-1})} dz = a_j^{-1} \ .$$

Figure 4: Construction of a differential form with prescribed residues.

This residue is the inverse of the desired linear part of the vector field. The metrics satisfy $g = \delta$.

Each cylindrical end p_j produces a simple zero of the vector field, and the cone point q one pole having order $-(s-2)$.

We leave to the interested reader the details when the polygon is degenerated.

Case 3. M is of genus $\mathsf{g} \geq 1$.

Consider on the sphere S^2 a meromorphic vector field Y with the given residues. Next we increase the genus of the two–manifold by consider the isometric connected sum of g copies of some torus.

In fact, given a flat torus $\mathbb{C}/\Lambda$, we cut it along a geodesic of finite g–length l and taking in (S^2, g) a geodesic of g–length l, which does not intersect the poles and zeros of the vector field Y, we cut S^2 along this geodesic. Finally, glue the cuts in the torus by isometries with the cuts in S^2. Note that the original vector field Y extends to the interior of the torus in a meromorphic fashion. Moreover, the end points in the cut give origin to two simple poles of the new vector field X in the connected sum $S^2 \cup \mathbb{C}/\Lambda$.

The case of higher genus is obvious, adding suitable copies of the torus. $\square$

Note that the above idea only produces some Riemann surfaces. We do not know if is it possible to obtain all the complex structures in the C^∞ manifold M, as is asserted in the classical proof, see [3] p. 52.

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