

Solvability of the general capacity problem in metric spaces *

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Abstract

Conditions are given under which the general capacity problem in metric spaces is solvable.

1991 Mathematics Subject Clasification: 90C08, 90C48

Keywords and phrases: General capacity problem; Optimization in spaces of measures.

1 Introduction

In the *general capacity* (GC) problem we are given the following data: (i) two metric spaces X and Y , endowed with the corresponding Borel σ -algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$; (ii) two nonnegative measurable functions $f : Y \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$, and (iii) a nonnegative measurable function $\varphi : X \times Y \rightarrow \mathbb{R}$. Then the GC problem can be stated as follows:

$$(1.1) \quad \mathbf{GC} \quad \text{minimize} \quad \langle \mu, f \rangle := \int_Y f(y) \mu(dy)$$

$$(1.2) \quad \text{subject to:} \quad \int_Y \varphi(x, y) \mu(dy) \geq g(x) \quad \forall x \in X, \quad \mu(\cdot) \geq 0.$$

Let \mathcal{F} be the class of *feasible solutions* for GC, that is, \mathcal{F} consists of the measures μ on Y that satisfy (1.2). In this paper we give conditions for the existence of an *optimal solution* μ^* for GC, so that μ^* is in \mathcal{F} and

$$(1.3) \quad \langle \mu^*, f \rangle = \inf \{ \langle \mu, f \rangle \mid \mu \in \mathcal{F} \}.$$

*Research partially supported by CONACYT and PROMEP scholarships.

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The capacity problem has its origin in electrostatics, where it is of interest to determine the capacity of a conducting body. As pointed out in [1, p.147] (see also [2]), the electrostatic capacity problem is related with potential theory, as in [5, 6]. The recognition of GC as an infinite-dimensional linear program was done in [21, 15]. An interesting connection between the general capacity problem and two-person zero-sum infinite games is mentioned in [1, 2]. Related studies appear in [12, 16, 20], although all of these references consider the case where X and Y are *compact* Hausdorff spaces.

It should be noted that our approach to the solvability of GC is quite straightforward. The underlying basic idea is the well-known fact that a lower semicontinuous function on a compact topological space attains its minimum value. (For a proof of this fact see, for instance, [3, p.389].) Our Assumption 2.1 is designed to give the appropriate meaning to “lower semicontinuity” and “compactness”. This assumption can also be used to construct suitable linear spaces of measures and functions on which the GC problem can be nicely formulated as a linear program. A similar linear programming (LP) formulation has been used in [7] to study the Monge–Kantorovich mass transfer problem in general metric spaces. An advantage of the LP formulation is that if the metric spaces in GC are separable, then the associated linear program can be approximated by *finite* linear programs [10].

2 Solvability of GC

We shall refer to the right-hand side of (1.3) as the *value* of the GC problem and denote it by $\inf(\text{GC})$. If GC is *solvable*, we then write the value $\inf(\text{GC})$ as $\min(\text{GC})$. In this case, (1.3) thus becomes

$$(2.1) \quad \langle \mu^*, f \rangle = \min(\text{GC}).$$

To prove the solvability of GC we shall require the following assumption.

Assumption 2.1 (a) *There exists a finite measure $\mu \in \mathcal{F}$ such that $\langle \mu, f \rangle < \infty$.*

(b) *The function f is inf-compact, which means that, for each $r \in \mathbb{R}$, the set*

$$(2.2) \quad K_r = \{y \in Y \mid f(y) \leq r\}$$

is compact. Moreover, f is strictly bounded away from zero, that is, there exists $\epsilon_0 > 0$ such that $f(y) \geq \epsilon_0$ for all $y \in Y$.

(c) For each $x \in X$, $\varphi(x, \cdot)$ is bounded above and upper semicontinuous (u.s.c.).

Observe that the *inf-compactness condition* implies that f is lower semicontinuous (l.s.c.). On the other hand, from the proof of Theorem 2.2 it can be seen that part (b) in Assumption 2.1 can be replaced by :

(b₁) $f(y)$ is strictly bounded away from zero and l.s.c.; and

(b₂) there exists a nondecreasing sequence of compact sets K_n in Y such that $K_n \uparrow Y$ and

$$\lim_{n \rightarrow \infty} \inf \{f(y) \mid y \notin K_n\} = \infty.$$

A function f that satisfies (b₂) is found in the literature under several different names: “moment” [8] or “strictly unbounded” [9, 11] or “Lyapunov” [13] or “norm-like” [14] function.

Let $M_+(Y)$ be the family of finite (nonnegative) measures on Y such that $\langle \mu, f \rangle < \infty$. By Assumption 2.1(a), $M_+(Y)$ is nonempty. We can now state our main result.

Theorem 2.2 *Under Assumption 2.1, the GC problem is solvable, that is, there exists a measure μ^* in $M_+(Y)$ that satisfies (1.2) and (2.1).*

Before proving Theorem 2.2 we shall introduce some concepts and preliminary results.

Definition 2.3 (See [4, 17, 18, 19].) *Let Γ be a family of finite measures on a metric space Y .*

(a) Γ is said to be *tight* if for each $\epsilon > 0$ there is a compact set K in Y such that $\mu(K^c) \leq \epsilon$ for all $\mu \in \Gamma$, where K^c denotes the complement of K .

(b) Γ is called *relatively compact* if for every sequence $\{\mu_n\}$ in Γ there exists a subsequence $\{\mu_m\}$ of $\{\mu_n\}$ and a finite measure μ on Y (not necessarily in Γ) such that $\{\mu_m\}$ converges weakly to μ , that is

$$(2.3) \quad \langle \mu_m, v \rangle \rightarrow \langle \mu, v \rangle \quad \forall v \in C_b(Y),$$

where $C_b(Y)$ stands for the space of continuous bounded functions on Y .

The statement ‘(b) implies (c)’ in the following lemma is part of Prohorov’s Theorem (see [4, 17, 18, 19]). The converse is also true if, for instance, Y is a complete and separable metric space (see [4, p.37], [17, p.47], [18, p.381]).

Lemma 2.4 *Suppose that f satisfies Assumption 2.1(b), and let Γ be a family of measures on Y . Consider the statements:*

- (a) *There exists a constant b such that $\int_Y f d\mu \leq b$ for all $\mu \in \Gamma$.*
- (b) *Γ is tight.*
- (c) *Γ is relatively compact.*

Then

$$(2.4) \quad (a) \implies (b) \implies (c).$$

Proof. As was already noted, the implication “(b) \implies (c)” is (part of) Prohorov’s Theorem. To prove that (a) implies (b), for each $n = 1, 2, \dots$, let K_n be the compact set $K_n = \{y \in Y \mid f(y) \leq n\}$. As f is strictly bounded away from zero, Γ is bounded, that is there is a constant $M \geq 0$ such that $\mu(Y) \leq M$ for all $\mu \in \Gamma$. Now, for any measure μ in Γ

$$b \geq \int_Y h d\mu \geq \int_{K_n^c} h d\mu \geq n\mu(K_n^c).$$

That is, $\mu(K_n^c) \leq b/n$ for all n , which clearly gives (b). ■

Lemma 2.5 *Suppose that $h : Y \rightarrow \mathbb{R}$ is a nonnegative and l.s.c. function, and let $\{\mu_m\}$ be a sequence of measures on Y . If μ_m converges weakly to μ [in the sense of (2.3)], then*

$$(2.5) \quad \liminf \langle \mu_m, h \rangle \geq \langle \mu, h \rangle.$$

Proof. By the hypotheses on h , there exists a sequence $\{v_k\}$ in $C_b(Y)$ such that $v_k \uparrow h$ pointwise as $k \rightarrow \infty$. Thus,

$$\langle \mu_n, h \rangle \geq \langle \mu_n, v_k \rangle \quad \forall n, k,$$

and, therefore, by weak convergence,

$$\liminf \langle \mu_n, h \rangle \geq \liminf \langle \mu_n, v_k \rangle = \langle \mu, v_k \rangle \quad \forall k.$$

Finally, letting $k \rightarrow \infty$, monotone convergence gives (2.5). ■

Remark 2.6 *If $h : Y \rightarrow \mathbb{R}$ is u.s.c. and bounded above, then instead of (2.5) we get*

$$(2.6) \quad \limsup \langle \mu_m, h \rangle \leq \langle \mu, h \rangle.$$

We are ready for the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\{\mu_n\}$ in $\mathcal{F} \cap M_+(Y)$ be a minimizing sequence for the GC problem; that is, each μ_n satisfies (1.2) and

$$(2.7) \quad \langle \mu_n, f \rangle \downarrow \inf(\text{GC}).$$

Thus, given $\epsilon > 0$, there exists an integer $n(\epsilon)$ such that

$$\inf(\text{GC}) \leq \langle \mu_n, f \rangle \leq \inf(\text{GC}) + \epsilon \quad \forall n \geq n(\epsilon).$$

As f is bounded away from zero, the minimizing sequence $\{\mu_n\}$ is bounded.

Next, in Lemma 2.4(a) take $b := \inf(\text{GC}) + \epsilon$. Hence, by (2.4), the sequence $\Gamma := \{\mu_n, n \geq n(\epsilon)\}$ is relatively compact, and so there is a subsequence $\{\mu_m\}$ of Γ and a measure μ^* in $M_+(Y)$ such that

$$\langle \mu_m, v \rangle \rightarrow \langle \mu^*, v \rangle \quad \forall v \in C_b(Y).$$

Moreover, by (2.5) and (2.7),

$$\langle \mu^*, f \rangle = \inf(\text{GC}).$$

Therefore, to prove that μ^* is an optimal solution for GC it suffices to show that μ^* is a feasible solution for GC, that is, μ^* satisfies the inequality in (1.2). Now, by Assumption 2.1(c) and Remark 2.6 we get

$$\int_Y \varphi(x, y) \mu^*(dy) \geq \limsup_{m \rightarrow \infty} \int_Y \varphi(x, y) \mu_m(dy) \geq g(x) \quad \forall x \in X,$$

which implies that μ^* is in \mathcal{F} . This yields that μ^* is an optimal solution for GC. ■

Acknowledgement

The author wishes to thank Dr. Onesimo Hernández-Lerma for his very valuable comments and suggestions.

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