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The eta invariant, connective K-theory and the Gromov-Lawson-Rosenberg conjecture *

Egidio Barrera-Yañez [1]

Abstract

Let $\ell = 2^\nu \geq 2$. Let M be an even dimensional manifold with cyclic fundamental group $\mathbb{Z}_\ell$. Assume the universal cover $\tilde{M}$ is spin. We shall define $N(M) = (\tilde{M} \times \tilde{M})/\mathbb{Z}_{2\ell}$ and express the eta invariant of $N(M)$ in terms of the eta invariant of M. We use this computation to determine certain equivariant connective K theory groups and establish the Gromov-Lawson-Rosenberg conjecture for some special cases in the non-orientable setting.

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1 Introduction

We say that an orientable manifold M is a spin manifold if we can lift the transition functions of the tangent bundle TM from the special orthogonal group $SO(m)$ to $Spin(m)$. Let M be a closed spin manifold of dimension m. We shall assume that m is at least 5 to ensure that certain surgery arguments work; these arguments fail in lower dimensions. If g is a Riemannian metric on M, let $D(M, s, g)$ be the associated Dirac operator defined by the *spin* structure s. We define the $\hat{A}$ genus as follows:

1. If $m \equiv 0 \bmod 4$, decompose $D(M,s,g) = D^+(M,s,g) + D^-(M,s,g)$ and let $\hat{A}(M,s,g) := \dim\ker(D^+(M,s,g)) - \dim\ker(D^-(M,s,g)) \in \mathbb{Z}$; the $D^\pm$ are the chiral spin operators.

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[1] Investigador, CINVESTAV-IPN

2. If $m \equiv 1 \bmod 8$, let $\hat{A}(M,s,g) = \dim\ker(D(M,s,g)) \in \mathbb{Z}_2$.
3. If $m \equiv 2 \bmod 8$, let $\hat{A}(M,s,g) = \frac{1}{2}\dim\ker(D(M,s,g)) \in \mathbb{Z}_2$.
4. If $m \not\equiv 0,1,2,4 \bmod 8$, let $\hat{A}(M,s,g) = 0$.

One can use the Atiyah-Singer index theorem to show $\hat{A}(M,s) = \hat{A}(M,s,g)$ is independent of the metric g. If M is simply connected, the spin structure s is unique and we let $\hat{A}(M) = \hat{A}(M,s)$.

If M admits a metric of positive scalar curvature, the formula of Lichnerowicz [16] shows there are no harmonic spinors; consequently $\hat{A}(M,s) = 0$. In other words, if there exists a spin structure s on M so that $\hat{A}(M,s) \neq 0$, then M does not admit a metric of positive scalar curvature. Gromov and Lawson conjectured that the $\hat{A}$ genus might be the only obstruction to the existence of a metric of positive scalar curvature if the dimension m was at least 5 and if M was a simply connected spin manifold. Stolz [24] established this conjecture by proving:

Theorem 1.0.1 *If M is a simply connected, closed, spin manifold of dimension $m \geq 5$, then M admits a metric of positive scalar curvature if and only if $\hat{A}(M) = 0$.*

The situation in the non-simply connected setting is quite different. Rosenberg has modified the original conjecture of Lawson and Gromov. Fix a group π. Let M be a connected manifold of dimension $m \geq 5$ with fundamental group π and *spin* universal cover. Rosenberg conjectured that M admits a metric of positive scalar curvature if and only if a generalized equivariant index α_π of the Dirac operator vanishes. For the fundamental groups that we shall be considering, α_π can be expressed in terms of the $\hat{A}$-genus defined above, see 2.8 for details. Kwasik and Schultz [14] showed the Gromov-Lawson-Rosenberg conjecture holds for a finite group π if and only if the conjecture holds for all the Sylow subgroups of π. Thus one can work one prime at a time. The Gromov-Lawson-Rosenberg conjecture has been established in the following cases:

1. If π is a spherical space form group and if M is *spin* (Botvinnik, Gilkey & Stolz [7]).
2. If π is cyclic and if M admits a flat *spin*^{c} structure (Botvinnik & Gilkey [5] and Kwasik & Schultz [15]).

3. If $\pi = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and if p is an odd prime (Schultz [23]).

4. If π belongs to a short list of infinite fundamental groups including free groups, free abelian groups and fundamental groups of orientable surfaces (Rosenberg & Stolz [20]).

Note that Schick [21] has shown that this conjecture fails in some instances so it is crucial to investigate the precise conditions under which the $\hat{A}$ genus carries the full set of obstructions.

The interesting phenomena in the papers cited above are in odd dimensions and the manifolds in question are orientable *spin*^{c} manifolds. In this paper, by contrast, we shall be primarily interested in non-orientable even dimensional *pin* and *pin*^{c} manifolds. Here is a brief guide to the paper. In Section 2, we present the necessary analytical preliminaries. Henceforth let $\ell = 2^\nu \geq 2$ and let $\mathbb{Z}_\ell$ be the cyclic group of order ℓ. Let M be a manifold of dimension m with fundamental group $\mathbb{Z}_\ell$ and *spin* universal cover $\tilde{M}$. If m is even, we assume M non-orientable; if m is odd, we assume M orientable. In 2.3, we define a twisted product $N(M)$ of dimension $2m$ associated to M which is non-orientable and which has a canonical *pin*^{c} structure. Our fundamental analytic result, stated in Theorem 2.3.2, relates the eta invariants of $N(M)$ and M if m is even; this generalizes previous work of Gilkey [10] relating the eta invariants of $N(M)$ and M if m is odd. In §2 we recall results concerning Clifford algebras, introduce the equivariant eta invariant, and prove Theorem 2.3.2. In §3, we shall use the results of §2 to compute the additive structure of some equivariant connective K-theory groups; see Theorem 4.0.3 for details. In §4, we shall prove the Gromov-Lawson-Rosenberg conjecture for certain non-orientable manifolds; see Theorem 5.0.5 for details.

2 Analytic Preliminaries

2.1 Notational conventions

Let $g_\ell = e^{2\pi\sqrt{-1}/\ell}$ be the canonical generator of the cyclic group $\mathbb{Z}_\ell = \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$ where $\ell = 2^\nu \geq 2$. Let $\rho_s(\lambda) := \lambda^s$; the ρ_s for s in the Poincare dual $\mathbb{Z}_\ell^* := \mathbb{Z}/\ell\mathbb{Z}$ parametrize the irreducible unitary representations of $\mathbb{Z}_\ell$. Let $\mathcal{P}$ be the universal principal $\mathbb{Z}_\ell$ bundle over the classifying space $B\mathbb{Z}_\ell$. A $mathbb_\ell$ structure on a manifold M is a map $f : M \mapsto B\mathbb{Z}_\ell$ is a $\mathbb{Z}_\ell$ structure on a manifold M, let $\mathcal{P}(M) := f^*\mathcal{P}$ be the associated $\mathbb{Z}_\ell$ principal bundle over M. If $\pi_1(M) = \mathbb{Z}_\ell$ and if f

defines the canonical $\mathbb{Z}_\ell$ structure over M, then $\mathcal{P}(M)$ is the universal cover $\tilde{M}$ of M.

2.2 The eta invariant

Let M be a compact Riemannian manifold of dimension m. If m is even, we assume that M admits a pin^c structure s; if m is odd, we assume that M admits a $spin^c$ structure s; see 3 and Gilkey [9] for further details. We assume M is equipped with a $\mathbb{Z}_\ell$ structure f. If ρ is a representation of $\mathbb{Z}_\ell$, let P_ρ be the Dirac operator defined by a metric g and the $(s)pin^c$ structure s with coefficients in the locally flat bundle of spinors Δ_ρ over M defined by ρ. The operator P_ρ is of Dirac type; there is a discrete spectral resolution $L^2(\Delta_\rho) = \oplus_\lambda E(\lambda, P_\rho)$. The eigenspaces $E(\lambda, P_\rho)$ are finite dimensional subspaces of the $C^\infty(\Delta_\rho)$. Let $\{\lambda_n\}$ denote the eigenvalues of P_ρ where each eigenvalue is repeated according to multiplicity; we have $|\lambda_n| \to \infty$. The eta function is defined by

$$\begin{aligned}\eta(P_\rho)(z) &:= \textstyle\sum_{\lambda_n \neq 0} \mathrm{sign}(\lambda_n)|\lambda_n|^{-z} + \dim E(0, P_\rho) \\ &:= \textstyle\sum_{\lambda \neq 0} \dim(E(\lambda, P_\rho))\mathrm{sign}(\lambda)|\lambda|^{-z} + \dim E(0, P_\rho).\end{aligned} \tag{1}$$

This sum converges to define a holomorphic function of z for $\Re e(z) \gg 0$; it has a meromorphic extension to $\mathbb{C}$ which is regular at $z = 0$; see [9]. The eta invariant is a measure of the spectral asymmetry of P_ρ which is defined by

$$\eta(M, f, s)(\rho) := \frac{1}{2}\{\eta(P_\rho)(z)\}_{z=0}.$$

The eta invariant is additive with respect to direct sums and extends to the group representation ring $RU(\mathbb{Z}_\ell)$.

2.3 Twisted products

Let f be a $\mathbb{Z}_\ell$ structure on a compact connected Riemannian manifold of dimension m. If m is even, assume that M is not orientable and that the line bundle $\rho_{\ell/2}(M)$ carries the orientation of M; if m is odd, we assume that M is orientable. Let $\mathcal{P}(M)$ be the associated $\mathbb{Z}_\ell$ principal bundle; $\mathcal{P}(M)$ is orientable. The action of the generator g_ℓ on $\mathcal{P}(M)$ reverses the orientation if m is even and preserves the orientation if m is odd. We assume that $\mathcal{P}(M)$ has a given *spin* structure and lift the action of g_ℓ to a morphism $\tilde{g}_\ell$ of the principal Pin^- bundle over $\mathcal{P}(M)$. Then $\tilde{g}_\ell^\ell$ covers the identity map of $\mathcal{P}(M)$ so $\tilde{g}_\ell^\ell = \pm 1$. If $\tilde{g}_\ell^\ell = +1$, then M admits a $(s)pin^-$ structure s_M so that the associated complex line bundle $\det(s_M)$ is trivial; if $\tilde{g}_\ell^\ell = -1$, then M admits a $(s)pin^c$ structure s_M so that the associated complex line bundle $\det(s_M)$ is given by the representation ρ_1. Give $\mathcal{P}(M) \times \mathcal{P}(M)$ the product *spin* structure. We define a fixed point free action of $\mathbb{Z}_{2\ell}$ on $\mathcal{P}(M) \times \mathcal{P}(M)$ by $g_{2\ell} : (x, y) \to (g_\ell \cdot y, x)$. Let

$$N := N(M) := (\mathcal{P}(M) \times \mathcal{P}(M))/\mathbb{Z}_{2\ell} \tag{2}$$

be the resulting quotient manifold. If m of M is even, the flip $(x, y) \to (y, x)$ preserves orientation of $\mathcal{P}(M) \times \mathcal{P}(M)$. Since $\rho_{\ell/2}$ carries the orientation of M, g_ℓ reverses the orientation of $\mathcal{P}(M)$. If m is odd, the flip reverses the orientation and g_ℓ preserves the orientation. Consequently regardless of the parity of m, the map $g_{2\ell}$ reverses the orientation of $\mathcal{P}(M) \times \mathcal{P}(M)$ so N is not orientable. If m is odd, then Gilkey [10] showed N admits a canonical pin^c structure; we generalize this result to the even dimensional case in §2. If $\tilde{g}_\ell^\ell = 1$ let $b = 0$; if $\tilde{g}_\ell^\ell = -1$, let $b = 1$. Then $\det(s_M) = \rho_b$. If m is odd, Gilkey proved [10] that:

Theorem 2.3.1 *Let m be odd and let M and $N(M)$ be as above. If $m \equiv 3 \mod 4$, let $\beta = 0$; if $m \equiv 1 \mod 4$, let $\beta = \ell/2$.*

1. *If $u = 2v - b + \beta$, then $\eta(N)(\rho_u) = \eta(M)(\rho_v) + \eta(M)(\rho_{v-\ell/2})$ in $\mathbb{R}/\mathbb{Z}$.*
2. *If $u = 2v - b + \beta + 1$, then $\eta(N)(\rho_u) = 0$ in $\mathbb{R}/\mathbb{Z}$.*
3. *If there are no harmonic spinors on $\mathcal{P}(M)$, the equalities above hold in $\mathbb{R}$ not just $\mathbb{R}/\mathbb{Z}$.*

In section 2, we generalize this result to even dimensional twisted products:

Theorem 2.3.2 *Let m be even and let M and $N(M)$ be as above.*

1. *If $\ell = 2$, then we have:*
   (a) *If $u = 2s - b + m/2$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$ in $\mathbb{R}/\mathbb{Z}$.*
   (b) *If $u = 2s - b + 1 + m/2$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s)$ in $\mathbb{R}/\mathbb{Z}$.*
2. *If $\ell > 2$, then we have:*
   (a) *If $u = 2s - b + m/2 + \ell/4$, then $\eta(N)(\rho_u) = \eta(M)(\rho_s + \rho_{s+\ell/4})$ in $\mathbb{R}/\mathbb{Z}$.*

   (b) If $u = 2s - b + m/2 + \ell/4 + 1$, then $\eta(N)(\rho_u) = 0$ in $\mathbb{R}/\mathbb{Z}$.

3. *If there are no harmonic spinors on $\mathcal{P}(M)$, then these equalities hold in $\mathbb{R}$.*

2.4 Equivariant spin bordism

Let ξ be a real vector bundle over $B\mathbb{Z}_\ell$. The equivariant spin bordism groups $MSpin_m(B\mathbb{Z}_\ell, \xi)$ are equivalence classes of triples (M, f, s) where M is a closed manifold of dimension m which need not be connected, f is a $\mathbb{Z}_\ell$ structure on M, and s is a *spin* structure on $T(M) \oplus f^*\xi$; we define the relation $(M, f, s) \sim 0$ in $MSpin_m(B\mathbb{Z}_\ell, \xi)$ if there exists a compact manifold Y with boundary M so that the structures s and f extend over Y.

Let M be a manifold with $\pi_1(M) = \mathbb{Z}_\ell$ whose universal cover $\mathcal{P}(M)$ admits a *spin* structure. Let ρ be a representation of $\mathbb{Z}_\ell$. There exists $0 \le i \le 3$ and a suitable structure s so that $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_\ell, \xi_i)$. Only two Stiefel Whitney classes ω_i for $i = 1, 2$ of the twisting bundle ξ play a role. There are 4 cases to consider:

1. We have $\omega_1(\xi_0) = 0$ and $\omega_2(\xi_0) = 0$. We may take ξ_0 to be the trivial line bundle and identify $MSpin_m(B\mathbb{Z}_\ell, \xi_0)$ with the ordinary equivariant spin bordism groups $MSpin_m(B\mathbb{Z}_\ell)$; such a manifold M admits a canonical *spin* structure s we use to define the eta invariant $\eta(M, f, s)(\rho)$ if m is odd.

2. We have $\omega_1(\xi_1) = 0$ and $\omega_2(\xi_1) \neq 0$. We may take ξ_1 to be the underlying real 2 plane bundle of the complex line bundle defined by the representation ρ_1. Such a manifold M admits a canonical *spin^c* structure s with determinant line bundle given by ρ_1 we use to define the eta invariant $\eta(M, f, s)(\rho)$ if m is odd.

3. We have $\omega_1(\xi_2) \neq 0$ and $\omega_2(\xi_2) = 0$. We may take ξ_2 to be the real line bundle defined by $\rho_{\ell/2}$. Such a manifold M admits a canonical *pin* structure s we use to define the eta invariant $\eta(M, f, s)(\rho)$ if m is even.

4. We have $\omega_1(\xi_3) \neq 0$ and $\omega_2(\xi_3) \neq 0$. We may take $\xi_3 = \xi_1 \oplus \xi_2$. Such a manifold M admits a canonical *pin^c* structure s with determinant line bundle given by ρ_1 we use to define the eta invariant $\eta(M, f, s)(\rho)$ if m is even.

2.5 Equivariant connective K-theory

Let $Thom(\xi)$ be the associated Thom space of the k dimensional real vector bundle ξ over $B\mathbb{Z}_\ell$. The equivariant connective K-theory groups are defined by

$$ko_m(B\mathbb{Z}_\ell,\xi)=\widetilde{ko}_{m+k}(Thom(\xi)).$$

Let $\mathbb{HP}^2$ be the quaternion projective space with the usual homogeneous metric of positive scalar curvature. Let $\mathbb{HP}^2\to E\to B$ be a fiber bundle where the transition functions are the group of isometries PSp^3 of $\mathbb{HP}^2$. Since $\mathbb{HP}^2$ is simply connected, the projection $p:E\to B$ induces an isomorphism on the fundamental group; any $\mathbb{Z}_\ell$ structure on E arises from a $\mathbb{Z}_\ell$ structure on B. Let $T_m(B\mathbb{Z}_\ell,\xi)$ be the subgroup of $MSpin_m(B\mathbb{Z}_\ell,\xi)$ generated by the total spaces E of geometric fibrations with fiber $\mathbb{HP}^2$. Stolz [25] has given the following geometrical characterization of these groups localized at the prime 2:

$$ko_m(B\mathbb{Z}_\ell,\xi)_{(2)}=\{MSpin_m(B\mathbb{Z}_\ell,\xi)/T_m(B\mathbb{Z}_\ell,\xi)\}_{(2)}.$$

The reduced groups $\widetilde{ko}_m(B\mathbb{Z}_\ell,\xi)$ are torsion 2-groups so it is not necessary to localize at the prime 2.

2.6 The eta invariant, bordism, and K-theory

As noted above, we can define the eta invariant $\eta(M,f,s)(\rho)$ if m is odd and if $\xi=\xi_0,\xi_1$. We can also define the eta invariant if m is even and if $\xi=\xi_2,\xi_3$. The eta invariant extends to the equivariant spin bordism groups $MSpin_m(\mathbb{Z}_\ell,\xi)$ and to the equivariant connective K-theory groups $ko_m(B\mathbb{Z}_\ell,\xi)$; these invariants are supported on the reduced bordism and K-theory groups. Let $RU_0(\mathbb{Z}_\ell)$ be the augmentation ideal of representations of virtual dimension 0. We refer to [5, 7, 10] for the proof of the following result.

Theorem 2.6.1 *Let $\rho\in RU(\mathbb{Z}_\ell)$.*

1. *Let m be odd and let $i=0,1$. If $m=4k+3$, assume $\rho\in RU_0(\mathbb{Z}_\ell)$. The map $M\to\eta(M,f,s)(\rho)$ extends to homomorphisms η_ρ from $MSpin_m(B\mathbb{Z}_\ell,\xi_i)$ and from $ko_m(B\mathbb{Z}_\ell,\xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m\equiv 3$ mod 8, if $i=0$, and if ρ is real, then we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.*

2. *Let m be even and let $i=2,3$. The map $M\to\eta(M,f,s)(\rho)$ extends to homomorphisms η_ρ from $MSpin_m(B\mathbb{Z}_\ell,\xi_i)$ and from*

$ko_m(B\mathbb{Z}_\ell, \xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 2 \bmod 8$, if $i = 2$, and if ρ is real, then we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.

2.7 Geometrical bordism groups

Let $\tau = \tau(g) := R_{ijji}$ be the scalar curvature of a Riemannian metric g; we consider quadruples (M, f, s, g) where (M, f, s) are as in 2.4, and where g is a metric of positive scalar curvature on M; necessarily $m \geq 2$. The geometric equivariant spin bordism groups ${}^{+}MSpin_m(B\mathbb{Z}_\ell, \xi)$ are defined similarly to the bordism groups $MSpin(B\mathbb{Z}_\ell, \xi)$. For $m \geq 2$, we consider quadruples (M, f, s, g) as above and say that $(M, f, s, g) \sim 0$ if there exists a compact manifold Y with boundary M so that the structures s and f extend over Y and so that the metric g extends over Y as a metric of positive scalar curvature which is the product near the boundary M. The forgetful functor defines a natural homomorphism from ${}^{+}MSpin_m(B\mathbb{Z}_\ell, \xi)$ to $MSpin_m(B\mathbb{Z}_\ell, \xi)$. Let $MSpin_m^+(B\mathbb{Z}_\ell, \xi)$ be the image of ${}^{+}MSpin_m(B\mathbb{Z}_\ell, \xi)$ under the forgetful functor. The elements of $T_m(B\mathbb{Z}_\ell, \xi)$ admit metrics of positive scalar curvature, see [7] for details. Let $ko_m^+(B\mathbb{Z}_\ell, \xi)$ be the image of $MSpin_m^+(B\mathbb{Z}_\ell, \xi)$ in $ko_m(B\mathbb{Z}_\ell, \xi)$.

2.8 The invariant α_π

Let $i = 0, 2$ and let $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_\ell, \xi_i)$. We express the invariant α_π of Rosenberg [18] in terms of the $\hat{A}$ genus as follows:

1. If $m \equiv 0 \bmod 4$, let $\alpha_\pi(M, f, s) := \hat{A}(\mathcal{P}(M)) \in \mathbb{Z}$.
2. If $\xi = \xi_0$ and if $m \equiv 1, 2 \bmod 8$, let s_L be the spin structure s twisted by the representation $\rho_{\ell/2}$. Let $\alpha_\pi(M, f, s) := \hat{A}(M, s) \oplus \hat{A}(M, s_L) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
3. If $\xi = \xi_2$ and if $m \equiv 2 \bmod 8$, let $\alpha_\pi(M, f, s) := \hat{A}(\mathcal{P}(M)) \in \mathbb{Z}$.
4. Set $\alpha_\pi(M, f, s) := 0$ otherwise.

Let $to_m(B\mathbb{Z}_\ell, \xi_i) := \widetilde{ko}_m(B\mathbb{Z}_\ell, \xi_i) \cap \ker(\alpha_\pi)$. Since α_π is surjective if $m \equiv 1 \bmod 8$ or $m \equiv 2 \bmod 8$, we may use α_π to construct the following short exact sequences:

$$\begin{aligned} &0 \to to_{8k+1}(B\mathbb{Z}_\ell) \to \widetilde{ko}_{8k+1}(B\mathbb{Z}_\ell) \to \mathbb{Z}_2 \to 0 \\ &0 \to to_{8k+2}(B\mathbb{Z}_\ell, \xi_2) \to \widetilde{ko}_{8k+2}(B\mathbb{Z}_\ell, \xi_2) \to \mathbb{Z}_2 \to 0 \end{aligned} \tag{3}$$

We remark that the first sequence in equation (3) splits if $k \geq 1$ and if $\ell = 4$ and that the second sequence in equation (3) splits if $k \geq 0$ for any ℓ; see [2] for details.

3 Clifford Algebras and Equivariant Eta Invariant

3.1 Clifford algebras

Let m be even throughout §2. Let $Clif^-(\mathbb{R}^m)$ be the real Clifford algebra; this is the universal unital algebra generated by $\mathbb{R}^m$ subject to the Clifford commutation relations $v*w+w*v = -2(v,w)$ for $v, w \in \mathbb{R}^m$. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis for $\mathbb{R}^m$. The orientation class is defined by

$$\omega_m = \sqrt{-1}^{m/2} e_1 * \cdots * e_m \in Clif^-(\mathbb{R}^m).$$

Since m is even, we have $\omega_m^2 = 1$. Let Δ_m be the *spin* representation. Clifford multiplication defines a map $c_m : \mathbb{R}^m \otimes_{\mathbb{R}} \Delta_m \to \Delta_m$ such that $c_m(\xi)^2 = -\|\xi\|^2$. We define the associated Clifford multiplication

$$\breve{c}_m(\xi) = \sqrt{-1} c_m(\omega_m) c_m(\xi). \tag{4}$$

Since ω_m anticommutes with ξ, $\breve{c}_m(\xi)^2 = -\|\xi\|^2$. Thus $\breve{c}_m$ also defines a representation of $Clif^-(\mathbb{R}^m)$ on Δ_m. Note that $\breve{c}_m(\omega_m) = c_m(\omega_m)$; also note that $c_m(\xi) = -\sqrt{-1}\breve{c}_m(\omega_m)\breve{c}_m(\xi)$. Let $Pin^-(m)$ be the multiplicative subgroup of $Clif^-(\mathbb{R}^m)$ which is generated by the unit sphere of R^m. Let $\chi : Pin^-(m) \to \mathbb{Z}_2$ be the orientation representation defined by $\chi(g) = \chi(v_1 * \cdots * v_k) = (-1)^k$;

$$c_m(\omega_m) c_m(g) = \chi(g) c_m(g) c_m(\omega_m).$$

Let $Spin(m) := ker(\chi) \cap Pin^-(m)$. If $g \in Spin(m)$, then $c_m(g) = \breve{c}_m(g)$. Let $\psi(g)(\xi) := \chi(g) g * \xi * g^{-1}$ define a representation from $Pin^-(m)$ to $O(m)$. The following diagram:

$$\begin{array}{ccc} \mathbb{R}^m \otimes \Delta_m & \xrightarrow{\breve{c}_m} & \Delta_m \\ \downarrow {\scriptstyle (\psi \otimes c_m)(g)} & & \downarrow {\scriptstyle c_m(g)} \\ \mathbb{R}^m \otimes \Delta_m & \xrightarrow{\breve{c}_m} & \Delta_m \end{array} \tag{5}$$

commutes because

$$\begin{aligned}&\breve{c}_m(\chi(g)g * \xi * g^{-1})c_m(g) = \sqrt{-1}c_m(\omega_m)c_m(\chi(g)g * \xi)\\ =&\sqrt{-1}c_m(g)c_m(\omega_m * \xi) = c_m(g)\breve{c}_m(\xi).\end{aligned}$$

Notice diagram (5) would not commute if we replaced $\breve{c}_m$ by c_m. This is the reason we introduced the auxiliary representation $\breve{c}_m$.

Let $\mathcal{Q}$ be the principal pin^c bundle over a pin^c manifold M. We complexify the representations ψ, c_m, and $\breve{c}_m$ to extend them to pin^c representations. The tangent bundle $TM := \mathcal{Q} \times_\psi \mathbb{R}^m$ is the bundle associated to $\mathcal{Q}$ by the representation ψ; the spinor bundle $\Delta_m(M) := \mathcal{Q} \times_{c_m} \Delta_m$ is the bundle associated to $\mathcal{Q}$ by the representation c_m. Diagram (5) shows that $\breve{c}_m$ extends to a linear map

$$\breve{c}_m : TM \otimes \Delta_m(M) \to \Delta_m(M) \text{ so } \breve{c}_m(\zeta)^2 = -|\zeta|^2.$$

Let ∇ be the spin connection on $\Delta_m(M)$. The Dirac operator discussed in §1 is defined by

$$P := \breve{c}_m \circ \nabla. \tag{6}$$

Let $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_\ell, \xi_i)$ for $i = 0, 1, 2, 3$ and let $\mathcal{P}(M)$ be the associated $\mathbb{Z}_\ell$ principal bundle; assume that $\mathcal{P}(M)$ has a *spin* structure. Let g_ℓ generate $\mathbb{Z}_\ell$; g_ℓ acts by isometries on $\mathcal{P}(M)$. Lift g_ℓ to an action $\tilde{g}_\ell$ on the principal Pin bundle of $\mathcal{P}(M)$ and let B_g be the associated action on the bundle $\Delta_m(\mathcal{P}(M))$ defined by c_m which covers the map g_ℓ. Let Q be the Dirac operator on $\Delta_m(\mathcal{P}(M))$ and let P be the Dirac operator on $\Delta_{2m}(\mathcal{P}(M) \times \mathcal{P}(M))$.

Lemma 3.1.1 *With the notation established above, we have:*

1. $c_m(\omega_m)Q = -Qc_m(\omega_m)$, $B_g c_m(\omega_m) = \psi(g)c_m(\omega_m)B_g$, *and* $B_g Q = QB_g$.
2. *Let* $\zeta = \xi \oplus \tilde{\xi} \in \mathbb{R}^m \oplus \mathbb{R}^m = \mathbb{R}^{2m}$. *Then:*
   (a) $\breve{c}_{2m}(\xi \oplus \tilde{\xi}) = c_m(\xi) \otimes 1 + c_m(\omega_m) \otimes c_m(\tilde{\xi})$.
   (b) $\breve{c}_{2m}(\omega_{2m}) = c_m(\omega_m) \otimes c_m(\omega_m)$.
   (c) $c_{2m}(\xi \oplus \tilde{\xi}) := -\sqrt{-1}\breve{c}_{2m}(\omega_{2m})\breve{c}_{2m}(\xi \oplus \tilde{\xi})$.
3. $\Delta_{2m}(\mathcal{P}(M) \times \mathcal{P}(M)) = \Delta_m(\mathcal{P}(M)) \otimes \Delta_m(\mathcal{P}(M))$.
4. $P = Q \otimes 1 + c_m(\omega_m) \otimes Q$.

Proof: The first assertion is immediate. We use (2-a) to define $\breve{c}_{2m}$. Because m is even, $\breve{c}_{2m}(\zeta)^2 = -|\zeta^2|$. Thus for dimensional reasons we may take $\breve{c}_{2m}$ to be the fundamental representation of the Clifford algebra; the remaining assertions now follow.

Let $\mathcal{B}_\ell$ and $\mathcal{B}_{2\ell}$ be the induced actions of g_ℓ and $g_{2\ell}$ on the bundles $\Delta_m(\mathcal{P}(M))$ and $\Delta_{2m}(\mathcal{P}(M)\times\mathcal{P}(M))$. Decompose $\Delta_m(\mathcal{P}(M)) = \Delta_m^+(\mathcal{P}(M)) \oplus \Delta_m^-(\mathcal{P}(M))$ into the chiral spin bundles, i.e. into the ± 1 eigenspaces of $c_m(\omega_m)$. Let $T(x,y) = (y,x)$. Let $\mathcal{B}_T$ be the action of T on $\Delta_{2m}(\mathcal{P}(M)\times\mathcal{P}(M))$. Define the map α by setting

$$
\begin{aligned}
\alpha(a_+\otimes b_+)(x,y) =& b_+(y,x)\otimes a_+(y,x)\\
\alpha(a_+\otimes b_-)(x,y) =& b_-(y,x)\otimes a_+(y,x)\\
\alpha(a_-\otimes b_+)(x,y) =& b_+(y,x)\otimes a_-(y,x)\\
\alpha(a_-\otimes b_-)(x,y) =& b_-(y,x)\otimes a_-(y,x).
\end{aligned}
\tag{7}
$$

We refer to [3] for details of the following Lemma.

Lemma 3.1.2 *With the notation established above, we have:*

1. $\mathcal{B}_T = \sqrt{-1}^{m/2}\alpha$.
2. $\mathcal{B}_{2\ell} = (\mathcal{B}_\ell \otimes c_m(\omega_m))\circ\mathcal{B}_T$.
3. *If* $\ell = 2$ *then* $[N(M)] \in MSpin_{2m}(B\mathbb{Z}_4, \xi_3)$.
4. *If* $\ell \geq 4$ *then* $[N(M)] \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, \xi_2)$.

3.2 Equivariant eta invariant

It is convenient to give a different formulation of Theorem 2.3.2. Let M be a closed connected non-orientable manifold of even dimension m with $\pi_1(M) = \mathbb{Z}_\ell$. Let $\mathcal{P}(M)$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on M. We set $b = 0$ if M admits a pin^- structure s and $b = 1$ if M admits a pin^c structure s. Then ρ_b defines the determinant line bundle of the structure s on M. Define the equivariant eta invariant:

$$\tilde{\eta}(M) := \sum_t \eta(M)(\rho_t)\otimes_{\mathbb{Z}}\rho_t \in \mathbb{R}\otimes_{\mathbb{Z}} RU(\mathbb{Z}_\ell). \tag{8}$$

If $h\in\mathbb{Z}_\ell$ and if $\tau\in RU(\mathbb{Z}_\ell)$, then we define $\tau(h) = \mathrm{Tr}\tau(h)\in\mathbb{C}$; this extends to an evaluation $\tilde{\eta}(M)(h)$ taking values in $\mathbb{C}$. Define $\tau_{2\ell}:\mathbb{Z}_\ell\to$

$U(2)$ by

$$
\begin{aligned}
&\tau_4 = \rho_{-b+m/2}(\rho_0 \oplus \rho_1) \text{ and } \tau_{2\ell} = \rho_{-b+m\ell/4}(\rho_{\ell/4} \oplus \rho_{-\ell/4}), \text{ i.e.}\\
&\tau_4(g_4) = \sqrt{-1}^{m/2} e^{-2\pi\sqrt{-1}b/4} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \text{ if } \ell = 2, \text{ and}\\
&\tau_{2\ell}(g_{2\ell}) = \sqrt{-1}^{m/2} e^{-2\pi\sqrt{-1}b/2\ell} \begin{pmatrix} e^{2\pi\sqrt{-1}/8} & 0 \\ 0 & e^{-2\pi\sqrt{-1}/8} \end{pmatrix} \text{ if } \ell \geq 4.
\end{aligned} \tag{9}
$$

Let $r : \mathbb{Z}_{2\ell} \to \mathbb{Z}_\ell$ be reduction mod ℓ; $r(g_{2\ell}) = g_\ell$. The dual map r^* from $RU(\mathbb{Z}_\ell)$ to $RU(\mathbb{Z}_{2\ell})$ is defined by $r^*(\rho_s) = \rho_{2s}$. Theorem 2.3.2 is equivalent to the following result. We refer to [3] for the proof of the following Theorem.

Theorem 3.2.1 *We have $\tilde{\eta}(N) = r^*(\tilde{\eta}(M)) \cdot Tr(\tau)$ in $\mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} RU(\mathbb{Z}_{2\ell})$. If there are no harmonic spinors on $\mathcal{P}(M)$, the equality holds in $\mathbb{R} \otimes_{\mathbb{Z}} RU(\mathbb{Z}_{2\ell})$.*

The following is an immediate consequence of Theorem 2.3.1, Theorem 3.2.1, and calculations of Gilkey [9]

Corollary 3.2.2 *For the projective spaces we have:*

1. $\eta(N(\mathbb{RP}^{4k}))(\rho_{2s+2k}) = \eta(\mathbb{RP}^{4k})(\rho_s) = (-1)^s 2^{-2k-1}$.
2. $\eta(N(\mathbb{RP}^{4k+1}))(\rho_{2s}) = \eta(\mathbb{RP}^{4k+1})(\rho_s(\rho_0 - \rho_1)) = (-1)^s 2^{-2k-1}$.
3. $\eta(N(\mathbb{RP}^{4k+2}))(\rho_{2s+2k-1}) = \eta(\mathbb{RP}^{4k+2})(\rho_s) = (-1)^s 2^{-2k-2}$.
4. $\eta(N(\mathbb{RP}^{4k+3}))(\rho_{2s}) = \eta(\mathbb{RP}^{4k+3})(\rho_s(\rho_0 - \rho_1)) = (-1)^s 2^{-2k-2}$.

4 Connective K-theory

Recall we defined $to_m(B\mathbb{Z}_4, \xi_i) := ko_m(B\mathbb{Z}_4, \xi_i) \cap \ker(\alpha_\pi)$ for $i = 0, 2$. We use the results of §2 to compute the following connective K-theory groups. Our results do not suffice to determine the additive structure of certain groups; these are marked by $\star$.

Theorem 4.0.3 *Let $k \geq 1$. We have:*

	$\widetilde{ko}_m(BZ_4)$	$to_m(BZ_4)$	$ko_m(BZ_4,\xi_1)$
$m=8k+1$	$\mathbb{Z}_{2^{2k+1}}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+3}}\oplus\mathbb{Z}_{2^{2k}}$
$m=8k+3$	$\mathbb{Z}_{2^{4k+3}}\oplus\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+3}}\oplus\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{2k+1}}$
$m=8k+5$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{4k+5}}\oplus\mathbb{Z}_{2^{2k+1}}$
$m=8k+7$	$\mathbb{Z}_{2^{4k+5}}\oplus\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{4k+5}}\oplus\mathbb{Z}_{2^{2k+1}}$	$\mathbb{Z}_{2^{2k+2}}$
	$ko_m(BZ_4,\xi_2)$	$to_m(BZ_4,\xi_2)$	$ko_m(BZ_4,\xi_3)$
$m=8k$	$\star$	$\star$	$\mathbb{Z}_{2^{2k+1}}$
$m=8k+2$	$\mathbb{Z}_{2^{2k+2}}\oplus\mathbb{Z}_2$	$\mathbb{Z}_{2^{2k+2}}$	$\star$
$m=8k+4$	$\star$	$\star$	$\mathbb{Z}_{2^{2k+2}}$
$m=8k+6$	$\mathbb{Z}_{2^{2k+2}}$	$\mathbb{Z}_{2^{2k+2}}$	$\star$

Before proving Theorem 4.0.3, we must establish some technical results. Suppose that $\vec{a}=(a_1,\dots,a_k)$ is a collection of odd integers. Let $\tau=\tau(\vec{a}):=\rho_{a_1}\oplus\cdots\oplus\rho_{a_k}$ define a fixed point free representation of $\mathbb{Z}_\ell$ in $U(k)$. The lens space associated with this representation is:

$$L^{2k-1}(\ell;\vec{a}):=S^{2k-1}/\tau(\mathbb{Z}_\ell).$$

Let $H^{\otimes 2}$ be the tensor square of the Hopf line bundle over $\mathbb{CP}^1$ which we identify with S^2. Let $\beta_k=H^{\otimes 2}\oplus(k-1)1_{\mathbb{C}}$ be a complex vector bundle of fiber dimension k over $\mathbb{CP}^1$. Let $S(\beta_k)$ be the associated sphere bundle, extend τ to a fixed point free action on $S(\beta_k)$. Let

$$X^{2k+1}(\ell;\vec{a}):=S(\beta_k)/\tau(\mathbb{Z}_\ell)$$

be the associated lens space bundle bundle over S^2. Both the lens spaces $L^{2k-1}(\ell;\vec{a})$ and the lens space bundles $X^{2k+1}(\ell;\vec{a})$ admit natural *spin*^{c} structures for $k\geq 2$; they are spin if and only if k is even. We refer to [7] for further details. Define:

1. If k is even, let $\mathcal{F}_L(\vec{a};\lambda)=\lambda^{-\|\vec{a}\|/2}\det(I-\tau(\lambda))$.
2. If k is odd, let $\mathcal{F}_L(\vec{a};\lambda)=\lambda^{-(\|\vec{a}\|+1)/2}\det(I-\tau(\lambda))$.
3. If $\lambda\neq 1$, let $\mathcal{G}_L(\vec{a};\lambda)=\mathcal{F}_L(\vec{a};\lambda)^{-1}$. If $\lambda=1$, let $\mathcal{G}_L(\vec{a};\lambda)=0$.
4. Let $\mathcal{G}_X(\vec{a};\lambda)=(1+\lambda^{a_1})(1-\lambda^{a_1})^{-1}\mathcal{G}_L(\vec{a};\lambda)$.

The following combinatorial formulas for the eta invariant of lens spaces and lens space bundles follow from work of Donnelly [8], see also [7].

Lemma 4.0.4 *We have:*

1. $\eta(L^{2k-1}(\ell;\vec{a}))(\rho) = \ell^{-1}\sum_{\lambda\in\mathbb{Z}_\ell,\lambda\neq 1} Tr(\rho)\mathcal{G}_L(\vec{a};\lambda) \in \mathbb{Q}$.

2. $\eta(X^{2k+1}(\ell;\vec{a}))(\rho) = \ell^{-1}\sum_{\lambda\in\mathbb{Z}_\ell,\lambda\neq 1} Tr(\rho)\mathcal{G}_X(\vec{a};\lambda) \in \mathbb{Q}$.

Let $\vec{a}_{2k} = (1,-1,\dots,1,-1)$. We have $L^{4k+3}(\cdot)$ and $X^{4k+1}(\cdot)$ are *spin*; we have $L^{4k+1}(\cdot)$ and $X^{4k+3}(\cdot)$ are $spin^c$. We define the following elements of equivariant connective K-theory.

1. $M_1 = M_1^{4k+3} = L^{4k+3}(4;\vec{a}_{2k},1,1) \in ko_{4k+3}(B\mathbb{Z}_4,\xi_0)$.
2. $M_2 = M_2^{4k+3} = L^{4k+3}(4;\vec{a}_{2k},1,3) \in ko_{4k+3}(B\mathbb{Z}_4,\xi_0)$.
3. $M_3 = M_3^{4k+1} = X^{4k+1}(4;\vec{a}_{2k-2},1,1) \in ko_{4k+1}(B\mathbb{Z}_4,\xi_0)$.
4. $N_1 = N_1^{4k+1} = L^{4k+1}(4;\vec{a}_{2k},1) \in ko_{4k+1}(B\mathbb{Z}_4,\xi_1)$.
5. $N_2 = N_2^{4k+1} = L^{4k+1}(4;\vec{a}_{2k},3) \in ko_{4k+1}(B\mathbb{Z}_4,\xi_1)$.
6. $N_3 = N_3^{4k+3} = X^{4k+3}(4;\vec{a}_{2k},1) \in ko_{4k+3}(B\mathbb{Z}_4,\xi_1)$.

By Theorem 2.6.1 $\eta(\cdot)(\rho)$ defines an $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}/2\mathbb{Z}$ valued invariant of $ko_*(B\mathbb{Z}_4,\xi_i)$ for $i = 0,1$. We use the formulas from Lemma 4.0.4 to compute the eta invariant of these manifolds.

	ρ_0	ρ_1	ρ_2	ρ_3
M_1	$-2^{-2k-4} - 2^{-k-2}$	2^{-2k-4}	$-2^{-2k-4} + 2^{-k-2}$	2^{-2k-4}
M_2	$2^{-2k-4} - 2^{-k-2}$	-2^{-2k-4}	$2^{-2k-4} + 2^{-k-2}$	-2^{-2k-4}
M_3	0	-2^{-k-1}	0	2^{-k-1}
N_1	$-2^{-2k-3} - 2^{-k-2}$	$2^{-2k-3} - 2^{-k-2}$	$-2^{-2k-3} + 2^{-k-2}$	$2^{-2k-3} + 2^{-k-2}$
N_2	$2^{-2k-3} - 2^{-k-2}$	$-2^{-2k-3} - 2^{-k-2}$	$2^{-2k-3} + 2^{-k-2}$	$-2^{-2k-3} + 2^{-k-2}$
N_3	-2^{-k-2}	2^{-k-2}	2^{-k-2}	-2^{-k-2}

Table A

We also use a computation of the orders of the equivariant connective K-theory groups by Botvinnik and Gilkey, see [5] for details of the following result:

	$\lvert\widetilde{ko}_m(B\mathbb{Z}_\ell,\xi_0)\rvert$	$\lvert ko_m(B\mathbb{Z}_\ell,\xi_1)\rvert$	$\lvert ko_m(B\mathbb{Z}_\ell,\xi_2)\rvert$	$\lvert ko_m(B\mathbb{Z}_\ell,\xi_3)\rvert$
$m = 8k$	2	1	2^{2k+1}	2^{2k+1}
$m = 8k+1$	$2(\ell/2)^{2k+1}$	$(2\ell)^{2k+1}$	2	1
$m = 8k+2$	2	1	2^{2k+3}	2^{2k+1}
$m = 8k+3$	$2(2\ell)^{2k+1}$	$(\ell/2)^{2k+1}$	2	1
$m = 8k+4$	1	1	2^{2k+2}	2^{2k+2}
$m = 8k+5$	$(\ell/2)^{2k+2}$	$(2\ell)^{2k+2}$	1	1
$m = 8k+6$	1	1	2^{2k+2}	2^{2k+2}
$m = 8k+7$	$(2\ell)^{2k+2}$	$(\ell/2)^{2k+2}$	1	1

Table B

Proof: [Proof of Theorem 4.0.3] Since the manifolds M_i and N_i admit metrics of positive scalar curvature, the $\hat{A}(M_i) = 0$ and $\hat{A}(N_i) = 0$ and these manifolds belong to to_m. We apply Gaussian elimination to Table A to determine the range of the eta invariant applied to these manifolds and to obtain a lower bound of the subgroups of the appropriate connective K theory groups which are spanned by these manifolds. We compare this lower bound with the upper bound given in Table B for $\ell = 4$ to establish the second assertion. The only difference between $to_m(B\mathbb{Z}_4, \xi_0)$ and $\widetilde{ko}_m(B\mathbb{Z}_4, \xi_0)$ is in dimension $m = 8k+1$; the extra factor of $\mathbb{Z}_2$ arises because the extension in equation (3) splits. This completes the proof of Theorem 4.0.3 for m odd.

The twisted products of real projective spaces are the non-orientable manifolds that we use to compute the equivariant connective K-theory groups $ko_m(B\mathbb{Z}_\ell, \xi_i)$ for $i = 2, 3$; $N(\mathbb{RP}^{2k}) \in ko^+_{4k}(B\mathbb{Z}_4, \xi_3)$ and $N(\mathbb{RP}^{2k+1}) \in ko^+_{4k+2}(B\mathbb{Z}_4, \xi_2)$. We use Corollary 3.2.2 to compute the eta invariant of the manifolds $N(\mathbb{RP}^j)$ and obtain a lower estimate of the order of the subgroup of to_m generated thereby. We use the upper estimate of the orders of the equivariant connective K-theory groups given in Table B for $\ell = 4$. This establishes the result for $to_m(B\mathbb{Z}_4, \xi_2)$ and $ko_m(B\mathbb{Z}_4, \xi_3)$. Since the short exact sequence in equation (3) splits we have the final result.

5 The Gromov-Lawson-Rosenberg conjecture

We can prove this conjecture for some special cases in the non-orientable setting.

Theorem 5.0.5 *Let M be a connected closed non-orientable manifold of dimension m with $\pi_1(M) = \mathbb{Z}_4$. Assume that M admits a flat pin^c structure.*

1. *If $m = 4k \geq 8$ and if $\omega_2(M) \neq 0$, then M admits a metric of positive scalar curvature.*

2. *If $m = 4k+2 \geq 6$ and if $\omega_2(M) = 0$, then M admits a metric of positive scalar curvature.*

Proof: We use results from [2, 5] to see that to prove the theorem, it suffices to show that $ko^+_m(B\mathbb{Z}_\ell, \xi) = ker(\alpha_\pi) \cap ko_m(B\mathbb{Z}_\ell, \xi)$. Recall that we have

$$
\begin{aligned}
(10) \qquad to_{8k+2}(B\mathbb{Z}_4, \xi_2) &:= ko_{8k+2}(B\mathbb{Z}_4, \xi_2) \cap ker(\hat{A}). \\
&= ko_{8k+2}(B\mathbb{Z}_4, \xi_2) \cap ker(\alpha_\pi).
\end{aligned}
$$

It suffices to show $ko^+_{8k+2}(B\mathbb{Z}_4, \xi_2) = to_{8k+2}(B\mathbb{Z}_4, \xi_2)$. This follows from Theorem 4.0.3. This proves the second assertion; the first one is similar.

Egidio Barrera Yañez
Departmento de Matemáticas,
CINVESTAV-IPN
Apartado Postal 14-740
07000 México D. F.
ebarrera@math.cinvestav.mx

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