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AN EXAMPLE OF A BOUNDED-IN-PROBABILITY, BUT NON-TIGHT MARKOV CHAIN *

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Abstract

This paper shows an example of a non-tight Markov chain which is bounded in probability.

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1 Introduction

This note concerns the notions of tightness and boundedness in probability of Markov chains, which are related to the "stability" of dynamical systems (see e.g. Bhatia 1970, p. 41, and Meyn 1992, p.145) as well as the weak (pre-)compactness of probability measures (Billingsley 1968, p.37).

It is easy to see that

(1) *tighness implies boundedness in probability;*

see Definition 3. In fact, under suitable topological assumptions, both concepts are equivalent (Proposition 1). However, in this note we give an example of a bounded-in-probability Markov chain which is not tight – in other words, in a general context, these concepts are *not* equivalent.

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2 Preliminaries

Throughout the following, $(X, \mathbb{B}_X)$ denotes a measurable space in which X is a topological space and $\mathbb{B}_X$ stands for the corresponding Borel σ-algebra. If B is a subset of X, we denote by $\bar{B}$ and B^c its closure and its complement, respectively.

In addition, $\Phi = \{\Phi_t, t = 0, 1, \ldots\}$ denotes a X-valued time-homogeneous Markov chain with transition kernel $P(x, B)$; that is,

$$P(x, B) := \text{Prob}(\Phi_{t+1} \in B | \Phi_t = x) \ \ \forall\, x \in X,\ B \in \mathbb{B}_X,\ t = 0, 1, \ldots.$$

The n-step transition probability is $P^n(x, B) := P(\Phi_n \in B | \Phi_0 = x)$.

Definition 1 *A probability measure μ on $(X, \mathbb{B}_X)$ is said to be tight if, for each $\epsilon > 0$, there exists a compact set $K = K(\epsilon)$ in X such that $\mu(K) > 1 - \epsilon$.*

For example, each probability measure on $(X, \mathbb{B}_X)$ is tight if X satisfies one of the following conditions (see Billingsley 1968, and Meyn 1992):

(1) X is σ-compact;

(2) X is a Polish (that is, complete separable metric) space;

(3) X is a locally compact separable metric space.

In analogy with Definition 1, for a *family* M of probability measures on X we have:

Definition 2 *M is tight if from every $\epsilon > 0$ there is a compact set $K = K(\epsilon)$ in X such that $\mu(K) \geq 1 - \epsilon$ for all μ in M.*

For the Markov chain Φ, the notion of tightness (and of boundedness in probability) is an extension of Definition 2.

Definition 3 *(a) The Markov chain Φ is tight if the sequence $\{P^n(x, \cdot), n = 1, 2, \ldots\}$ is tight for each state $x \in X$; that is, for each $x \in X$ and $\epsilon > 0$, there is a compact set $K = K(x, \epsilon)$ in X such that $P^n(x, K) \geq 1 - \epsilon$ for all $n = 1, 2, \ldots$, or, equivalently, $P^n(x, K^c) \leq \epsilon$ for all $n = 1, 2, \ldots$.*

(b) Φ is bounded in probability if for each $x \in X$ and $\epsilon > 0$ there is a compact set $K = K(x, \epsilon))$ in X such that $\liminf_{n\to\infty} P^n(x, K) \geq 1 - \epsilon$.

As was already noted in (1), it is obvious that if Φ is tight, then it is bounded in probability. Moreover we have:

Proposition 1 *Suppose that for each $x \in X$ and $n = 1, 2, \ldots$, the n-step transition probability $P^n(x, \cdot)$ is tight. Then the Markov chain Φ is tight if and only if it is bounded in probability.*

Proof: ($\Leftarrow$) Suppose that Φ is bounded in probability, and choose $x \in X$ and $\epsilon > 0$ arbitrary. Then there exist a compact set $K = K(x, \epsilon)$ and an integer $N = N(x, \epsilon)$ such that $P^n(x, K) \geq 1 - \epsilon$ for all $n \geq N$. On the other hand, for the tightness, for each $j = 1, \ldots, N-1$ there is a compact K_j such that $P^j(x, K_j) \geq 1 - \epsilon$. Therefore, the compact set $K_* := K_1 \cup \cdots \cup K_{N-1} \cup K$ satisfies that $P^n(x, K_*) \geq 1 - \epsilon$ for all $n = 1, 2, \ldots$. Thus, as $x \in X$ and $\epsilon > 0$ were arbitrary, Φ is tight.

The converse follows from (1). ∎

Each of the conditions (2) to (4) implies the hypothesis of Proposition 1, in which case tightness and boundedness-in-probability of Φ are equivalent. This is not necessarily true for a general topological space X, as the following example shows.

3 The example

Suppose that X is an uncountable set, and let us endowed it with the "countable complement" topology τ_X, which consists of the empty set $\emptyset$ and all the sets $B \subset X$ for which B^c is countable.

Proposition 2 *(a) If B is an infinite subset of X, then B is not compact.*

(b) The topological space (X, τ_X) is not locally compact.

Proof: (a) Let $B \subset X$ be an infinite set, and $A = \{a_1, a_2, \ldots\} \subset B$ a countable subset of B. Let $B_n := A^c \cup \{a_1, \ldots, a_n\}$, $n = 1, 2, \ldots$. Then $\{B_n\}$ is an open cover of B without a finite subcover.

(b) If $B \in \tau_X$ is nonempty, then $\bar{B} = X$, which, by (a), is not a compact set. ∎

Now let $\mu : \mathbb{B}_X \to [0, 1]$ be the set function defined as

$$\mu(B) = \begin{cases} 0 & \text{if } B \text{ is a countable set,} \\ 1 & \text{if } B^c \text{ is a countable set.} \end{cases}$$

Proposition 3 *μ is a non-tight probability measure.*

Proof: As μ is a nonnegative function and $\mu(X) = 1$, to prove that μ is a probability measure it suffices to show that it is σ-additive. To prove the latter, let $\{B_n\}$ be a sequence of disjoint sets in $\mathbb{B}_X$, and let $B := \bigcup_{n=1}^{\infty} B_n$. There are now two cases:

(i) All the B_n are countable sets;

(ii) One and only one of the sets B_n has a countable complement – there cannot exist two disjoint uncountable sets in $\mathbb{B}_X$.

In either case, we have that $\mu(B) = \sum_n \mu(B_n)$; that is μ is σ-additive.

Finally, observe that, by Proposition 2, the compact sets have μ-measure zero; hence μ is not tight. ∎

Proposition 3 suggests the following definition of a bounded-in-probability, but non-tight Markov chain.

Let a, b two points not in X and set $Y := X \cup D$, where $D := \{a, b\}$. Let $\mathbb{B}_D$ be the discrete σ-algebra on D, and $\mathbb{B} := \sigma\{\mathbb{B}_X \cup \mathbb{B}_D\}$ the associated σ-algebra on Y.

Proposition 4 *Consider a Markov chain on Y with transition kernel $\hat{P}$ given by:*

$$\begin{aligned}
&\hat{P}(a, \{a\}) := \hat{P}(a, \{b\}) := 0;\\
&\hat{P}(a, B) := \mu(B) \text{ if } B \in \mathbb{B}_X;\\
&\hat{P}(x, \{b\}) = 1 \text{ if } x \in X;\\
&\hat{P}(b, \{b\}) = 1.
\end{aligned}$$

Then the Markov chain, say Φ, associated to $\hat{P}$ is bounded in probability, but is not tight.

Proof: That Φ is not tight follows from Proposition 3, whereas boundedness in probability follows from the fact that $\hat{P}^n(a, \{b\}) = \mu(X) = 1$ for all $n \geq 2$. ∎

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