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CANCELLATION LAWS IN TOPOLOGICAL PRODUCTS

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Abstract

Given three spaces A, B and H such that $A \times H$ is homeomorphic to $B \times H$, when are A and B homeomorphic? In this paper we answer positively this old question when A and B are subsets of the real line and H is connected.

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1 Introduction

Given three spaces A, B and H such that $A \times H \cong B \times H$, when are A and B homeomorphic? By *space* we mean *non-empty topological space.* This is an old question in topology without a complete answer so far, see for example [1], [2], [3], [4] and [7]. Results which give conditions for the answer to be *positive* are known as *cancellation laws.* In 1995 a notable cancellation law was given by Behrends and Pelant in Theorem 1 of [2]:

Let H be a compact connected Hausdorff space such that the only continuous mappings from H to itself are the constant ones and the identity; then for arbitrary (compact) Hausdorff spaces A and B, the cancellation law $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds.

The main purpose of this paper is to present several cancellation laws all of which are deduced from Theorems 1.1 and 1.2 (proved in sections 2 and 3 respectively). The spaces: $\mathbb{R}$, $\mathbb{R}^+ = [0, \infty) \subseteq \mathbb{R}$ and $\mathcal{I} = [0, 1] \subseteq \mathbb{R}$ are respectively the real line, the semi-closed interval

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and the closed unit interval of the real line; they are all endowed with the standard topology. In this paper we show the following results.

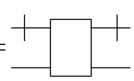
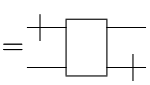
Theorem 1.1 *Let H be a connected space such that the following four spaces are not homeomorphic by pairs: $\mathbb{R} \times H$, $\mathbb{R}^+ \times H$, $\mathcal{I} \times H$ and H. Then the cancellation law: $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds when A and B are arbitrary subsets of the real line.*

Theorem 1.2 *Let H be a pathwise connected space. Then the cancellation law: $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds for all Hausdorff spaces A and B which both contain no copies of the real line.*

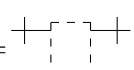
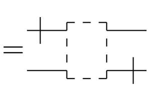
Likewise, there exist spaces which do not satisfy the cancellation law.

Example 1.3 No two of the intervals $\mathbb{R}$, $\mathbb{R}^+$ and $\mathcal{I}$ are homeomorphic; however it is easy to prove that the following three products are homeomorphic: $\mathbb{R} \times \mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^+$ and $\mathcal{I} \times \mathbb{R}^+$.

Example 1.4 The following two closed subsets $A, B \subseteq \mathbb{R}^2$ are clearly non-homeomorphic; however, $A \times \mathcal{I} \cong B \times \mathcal{I}$ (both subsets are built by adding "branches" to the closed square $\mathcal{I}^2$):

$A :=$  $B :=$ 

Moreover, by adding "branches" to an open square of $\mathbb{R}^2$, we also can build two non-homeomorphic subsets $A, B \subseteq \mathbb{R}^2$ such that $A \times \mathbb{R} \cong B \times \mathbb{R}$:

$A :=$  $B :=$ 

This paper is divided into four sections as follows: Section 1 is the introduction. Theorem 1.1 is shown in section 2. Sections 3 is devoted to prove Theorem 1.2. Applications of Theorems 1.1 and 1.2 are presented in section 4.

2 Proof of Theorem 1.1

Recall the hypotheses of Theorem 1.1: A and B are subsets of the real line, the space H is connected and the four spaces $\mathbb{R} \times H$, $\mathbb{R}^+ \times H$, $\mathcal{I} \times H$ and H are not homeomorphic by pairs. Now, in the cancellation law $A \times H \cong B \times H \Leftrightarrow A \cong B$, the implication $[\Leftarrow]$ is trivial. Whence we just need to show the converse implication $[\Rightarrow]$. Let g be a fixed homeomorphism from $A \times H$ onto $B \times H$.

If A and/or B are empty, we have immediately that A and B are homeomorphic. Hence, we shall suppose that A and B are both non-empty sets. Decompose them into their connected components (see [5, p. 111]):

$\{\mathcal{A}_k\}_{k\in K}$ are the connected components of A, and
$\{\mathcal{B}_l\}_{l\in L}$ are the connected components of B.

Since H is connected, we have immediately that (see Exemple 3 of [5,page 111]):

$\{\mathcal{A}_k \times H\}_{k\in K}$ are the connected components of $A \times H$, and
$\{\mathcal{B}_l \times H\}_{l\in L}$ are the connected components of $B \times H$.

It is easy to see that $A \times H$ and $B \times H$ have got the same number of connected components. Whence, we can assume that $L = K$; and without loss of generality, we likewise can suppose that $g(\mathcal{A}_k \times H) = \mathcal{B}_k \times H$ for every index $k \in K$. Observe the following facts (recall that $\mathrm{Bd}_A\mathcal{A}_k = \overline{\mathcal{A}_k} \cap \overline{A - \mathcal{A}_k}$), given $k \in K$:

i) $g(\mathrm{Bd}_A\mathcal{A}_k \times H) = g(\mathrm{Bd}_{A\times H}(\mathcal{A}_k \times H)) = \mathrm{Bd}_{B\times H}(\mathcal{B}_k \times H) = \mathrm{Bd}_B\mathcal{B}_k \times H$.

ii) $\mathcal{A}_k$ (and $\mathcal{B}_k$) is a singleton or it is homeomorphic to $\mathbb{R}$, $\mathbb{R}^+$ or $\mathcal{I}$.

iii) $\mathrm{Bd}_A\mathcal{A}_k \subseteq \mathcal{A}_k \subseteq \mathbb{R}$, because $\mathcal{A}_k$ is closed in A (see [5, p. 112]).

iv) When $\mathcal{A}_k$ is not a singleton, each point of $\mathrm{Bd}_A\mathcal{A}_k$ is an end point of $\mathcal{A}_k$, from points (ii) and (iii).

v) Likewise, when $\mathcal{B}_k$ is not a singleton, each point of $\mathrm{Bd}_B\mathcal{B}_k$ is an end point of $\mathcal{B}_k$.

vi) Finally: $|\mathrm{Bd}_A\mathcal{A}_k| = |\mathrm{Bd}_B\mathcal{B}_k| \leq 2$, by points (i), (iv) and (v).

From the facts above, we can conclude the following:

Lemma 2.1 *Let $k \in K$ be given, then for every $x \in Bd_A\mathcal{A}_k$ (resp. $y \in Bd_B\mathcal{B}_k$) there exists $y \in Bd_B\mathcal{B}_k$ (resp. $x \in Bd_A\mathcal{A}_k$) such that $g(\{x\} \times H) = \{y\} \times H$.*

On the other hand, from the given hypotheses we can deduce that $\mathcal{A}_k$ and $\mathcal{B}_k$ are homeomorphic for every index $k \in K$. Otherwise, if $\mathcal{A}_k \ncong \mathcal{B}_k$, we would have that $\mathcal{A}_k \times H \ncong \mathcal{B}_k \times H$, a contradiction to $g(\mathcal{A}_k \times H) = \mathcal{B}_k \times H$. Moreover, using Lemma 2.1, we can find a homeomorphism G_k from $\mathcal{A}_k$ onto $\mathcal{B}_k$ such that: $G_k(x) \times H = g(\{x\} \times H)$ for each $x \in \mathrm{Bd}_A\mathcal{A}_k$; and $G_k^{-1}(y) \times H = g^{-1}(\{y\} \times H)$ for each $y \in \mathrm{Bd}_B\mathcal{B}_k$. Now build the function $\Gamma : A \to B$ as follows:

$$\text{If } a \in \mathcal{A}_k \subseteq A \text{ for some } k \in K, \text{ then } \Gamma(a) = G_k(a).$$

Since $\{\mathcal{A}_k\}_{k\in K}$ is a "partition" of A and $\{\mathcal{B}_k\}_{k\in K}$ is a "partition" of B (see [5, p. 112]), it is easy to deduce that Γ is a bijective function. We assert that Γ is a homeomorphism from A onto B as well. Firstly, we show that Γ is a continuous mapping. Given an index $l \in K$ and a point $a \in A_l$, consider a sequence $\{a_n\}_{n\in \mathbb{N}}$ which converges to a when $n \to \infty$ (the set of natural numbers is denoted by $\mathbb{N}$). Without loss of generality, we need to analyze only three cases:

1) $\{a_n\}_{n\in \mathbb{N}} \subseteq \mathcal{A}_l$.
2) $\{a_n\}_{n\in \mathbb{N}} \subseteq A - \mathcal{A}_l$.
3) $\{a_n\}_{n\in \mathbb{N}} = \{b_n\}_{n\in \mathbb{N}} \cup \{c_n\}_{n\in \mathbb{N}}$ such that $\{b_n\}_{n\in \mathbb{N}} \subseteq A - \mathcal{A}_l$, $\{c_n\}_{n\in \mathbb{N}} \subseteq \mathcal{A}_l$, and both: $\{b_n\}_{n\in \mathbb{N}}$ and $\{c_n\}_{n\in \mathbb{N}}$ converge to a.

When case (1) holds, we have got that $\Gamma(a_n) = G_l(a_n) \subseteq \mathcal{B}_l$ for every $n \in \mathbb{N}$. Hence, the sequence $\{\Gamma(a_n)\}_{n\in \mathbb{N}}$ converges to $G_l(a) = \Gamma(a)$ when $n \to \infty$, because G_l is a homeomorphism.

In case (2), it is easy to note that $a \subseteq \mathrm{Bd}_A\mathcal{A}_l$. Now, for each $n \in \mathbb{N}$, let $\mathcal{A}_{k(n)}$ be the connected component which contains the point a_n (note that $\mathcal{A}_{k(n)} \neq \mathcal{A}_l$). By the structure of the real line, given an open ball $B(a, \varepsilon)$ of center a and radius $\varepsilon > 0$, there exist a natural number n_0 such that $\mathcal{A}_{k(n)} \subseteq B(a, \varepsilon)$ when $n \geq n_0$. That is: the sequence of sets $\{\mathcal{A}_{k(n)}\}_{n\in \mathbb{N}}$ converges to the singleton $\{a\}$ when $n \to \infty$. Likewise, we have got that $\{\mathcal{A}_{k(n)} \times H\}_{n\in \mathbb{N}}$ converges to $\{x\} \times H$. Now then, since $g(\mathcal{A}_{k(n)} \times H) = \mathcal{B}_{k(n)} \times H$ for each $n \in \mathbb{N}$, and g is a homeomorphism from $A \times H$ onto $B \times H$; there exists a point $y \in \mathrm{Bd}_B\mathcal{B}_l$ such that the sequence $\{\mathcal{B}_{k(n)} \times H\}_{n\in \mathbb{N}}$ converges to $\{y\} \times H$ when $n \to \infty$ and $g(\{a\} \times H) = \{y\} \times H$ (recall that $a \in \mathrm{Bd}_A\mathcal{A}_l$ and use Lemma 2.1). On the other hand, notice that $\Gamma(a_n) \in \Gamma(\mathcal{A}_{k(n)}) = \mathcal{B}_{k(n)}$ for each $n \in \mathbb{N}$.

Moreover, the equality $\Gamma(a) = G_k(a) = y$ holds (recall the definition of G_k). Whence, the sequence $\{\Gamma(a_n)\}_{n\in\mathbb{N}}$ converges to $\Gamma(a)$ when $n \to \infty$.

Finally, in case (3), we proceed as in cases (1) and (2). In both cases the sequences $\{\Gamma(b_n)\}_{n\in\mathbb{N}}$ and $\{\Gamma(c_n)\}_{n\in\mathbb{N}}$ converge to $\Gamma(a)$ when $n \to \infty$. Whence, their union $\{\Gamma(b_n)\}_{n\in\mathbb{N}} \cup \{\Gamma(c_n)\}_{n\in\mathbb{N}} = \{\Gamma(a_n)\}_{n\in\mathbb{N}}$ converges to $\Gamma(a)$ as well. Thus, Γ is a continuos mapping. In a similar way we can prove that Γ^{-1} is an open mapping, and so Γ is a homeomorphism from A onto B. ∎

3 Proof of Theorem 1.2

Recall the hypotheses of Theorem 1.2: A and B are Hausdorff spaces which both contain no copies of the real line and the space H is pathwise connected. Now, in the cancellation law $A \times H \cong B \times H \Leftrightarrow A \cong B$, the implication [$\Leftarrow$] is trivial. Thus, we just show the converse implication [$\Rightarrow$].

We assert that A (and B) is totally pathwise disconnected (see [5, p. 119]). Otherwise, if there exists a non-constant continuous mapping $f : \mathcal{I} \to A$, the image $f(\mathcal{I})$ would be a non-degenerate Peano continuum by the Hahn-Mazurkiewicz theorem (see [8, p. 298] and [9, p. 128]). Then, the set $f(\mathcal{I})$ would contain a copy of the real line (see [9, p. 130]), a contradiction. Likewise, the space B is totally pathwise disconnected.

Let g be a fixed homeomorphism from $A \times H$ onto $B \times H$. Since H is pathwise connected, we can deduce that $\{\{a\} \times H\}_{a\in A}$ (resp. $\{\{b\} \times H\}_{b\in B}$) are the pathwise connected components of $A \times H$ (resp. $B \times H$), see [6, p. 461]. Hence, for each $a \in A$ (resp. $b \in B$) there exists unique $b \in B$ (resp. $a \in A$) such that $g(\{a\} \times H) = \{b\} \times H$. We use just the last statement to show that A and B are homeomorphic, thus we can change the given hypotheses and use other hypotheses which assure that the last statement holds. For example, we can change pathwise connectedness to connectedness, σ-connectedness, etc.

Define the relation $R : A \to B$ by $R = \pi_B \circ g \circ \pi_A^{-1}$. Let $a \in A$ be a given point, from the preceding paragraph, we have got that $R(a) = \pi_B \circ g \circ \pi_A^{-1}(a) = \pi_B \circ g(\{a\} \times H) = \pi_B(\{b\} \times H) = b$ (where $b \in B$ is unique). Likewise, for every $b \in B$ there exists a unique point $a \in A$ such that $R(a) = b$. Therefore, R is a bijective function. On the other hand, π_A and π_B are open and continuous mappings, so R is also a open and continuous mapping. Whence $R : A \to B$ is a homeomorphism. ■

4 Applications

We now present applications of Theorems 1.1 and 1.2. There exist many spaces satisfying the hypotheses of Theorem 1.1, for instance:

Proposition 4.1 *Let H be a connected n-manifold without boundary (n is some natural number). Then the cancellation law: $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds for arbitrary subsets $A, B \subseteq \mathbb{R}$.*

Proof: Consider two manifolds X and Y. If we denote by ∂X the boundary of X as manifold (we use the notation ∂ in order to distinguish this from Bd, the boundary as space), then $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$. Therefore:

$$\begin{array}{ll} \partial H = \emptyset & \partial(\mathbb{R}^+ \times H) = \{0\} \times H \text{ is connected.} \\ \partial(\mathbb{R} \times H) = \emptyset & \partial(\mathcal{I} \times H) = \{0,1\} \times H \text{ is disconnected.} \end{array}$$

On the other hand, the dimension of the manifold $\mathbb{R} \times H$ is $n+1$. Whence, the four spaces: $\mathcal{I} \times H$, $\mathbb{R} \times H$, $\mathbb{R}^+ \times H$ and H are not homeomorphic by pairs. The result follows then from Theorem 1.1. ■

We can deduce more cancellation laws, using the following lemma due to professor Alejandro Illanes Mejía (private communication):

Lemma 4.2 *Given a connected compact space H, we have: $\mathbb{R}^+ \times H \ncong \mathbb{R} \times H$.*

Proof. Suppose that $\mathbb{R} \times H$ is homeomorphic to $\mathbb{R}^+ \times H$ and take $g : \mathbb{R}^+ \times H \to \mathbb{R} \times H$ a homeomorphism, $\pi_1 : \mathbb{R} \times H \to \mathbb{R}$ the projection, $\pi_2 : \mathbb{R}^+ \times H \to \mathbb{R}^+$ the projection and $E = \{1\} \times H \subseteq \mathbb{R} \times H$.

Since H is compact and π_2 is a continuous mapping, the set $\pi_2 \circ g^{-1}(E) \subseteq \mathbb{R}^+$ is compact, and so it is bounded. Whence, there exists a point $a \in \mathbb{R}^+$ such that $\pi_2 \circ g^{-1}(E) \subseteq [0, a]$. Besides, the inclusions $g^{-1}(E) \subseteq D \subseteq \mathbb{R}^+ \times H$ hold, where $D = [0, a] \times H$ is compact. Now note that $(\mathbb{R}^+ \times H) - D = (a, \infty) \times H$ is connected. Therefore, applying g we obtain that $E \subseteq g(D) \subseteq \mathbb{R} \times H$, where $g(D)$ is compact and $(\mathbb{R} \times H) - g(D)$ is connected.

On the other hand, the set $\pi_1 \circ g(D) \subseteq \mathbb{R}$ is compact and so it is bounded. That is, there exist two points $b, c \in \mathbb{R}$, $b < c$, such that $\pi_1 \circ g(D) \subseteq [b,c]$. Hence: $\{1\} \times H = E \subseteq g(D) \subseteq [b,c] \times H \subseteq \mathbb{R} \times H$. Taking complements in these inclusions (considering $\mathbb{R} \times H$ as the universe set), we obtain:

$$((-\infty, b) \times H) \cup ((c, \infty) \times H) \subseteq (\mathbb{R} \times H) - g(D) \subseteq (\mathbb{R} - \{1\}) \times H.$$

This implies that $\mathbb{R} \times H - g(D)$ is not connected, a contradiction. ∎

Proposition 4.3 *Let H be a connected compact space which is not homeomorphic to $\mathcal{I} \times H$ (for example, $H = \mathcal{I}$ or $H = \mathcal{S}^1$). Then the cancellation law: $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds for arbitrary subsets $A, B \subseteq \mathbb{R}$.*

Proof: It is easy to see that H and $\mathcal{I} \times H$ are compact, but $\mathbb{R} \times H$ and $\mathbb{R}^+ \times H$ are not. Hence, from the given hypotheses and Lemma 4.2, the four spaces: $\mathcal{I} \times H$, $\mathbb{R} \times H$, $\mathbb{R}^+ \times H$ and H are not homeomorphic by pairs. The result follows then from Theorem 1.1. ∎

We conclude this paper with the following three examples, which illustrate Theorems 1.1 and 1.2 (and Propositions 4.1 and 4.3).

Example 4.4 If we take $H = \mathbb{R}$, $H = \mathcal{I}$ or $H = S^1$, the *cancellation law* $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds for arbitrary subsets $A, B \subseteq \mathbb{R}$. However, if we take $H = \mathbb{R}^+$, this cancellation law fails to hold as example 1.3 shows. Moreover, if we take $A, B \subseteq \mathbb{R}^n$ with n a natural number greater than one, the cancellation law may fail to hold as Fox's example (see [7]) and example 1.4 show.

Example 4.5 Let X be a given Hausdorff space, the following two statements are equivalent:

1. X contains no copies of the real line.
2. For all $A, B \subseteq X$, *the cancellation law*: $A \times \mathbb{R}^+ \cong B \times \mathbb{R}^+ \Leftrightarrow A \cong B$ holds.

Indeed, (1) implies (2) by Theorem 1.2. In order to show that (2) implies (1), suppose that X contains a copy of $\mathbb{R}$. Then we can find two subsets $A, B \subseteq X$ which are respectively homeomorphic to $\mathbb{R}$ and $\mathcal{I}$. Hence: $A \ncong B$, but $A \times \mathbb{R}^+ \cong B \times \mathbb{R}^+$ according to Example 1.3, a contradiction.

Likewise, reasoning as in the last paragraph, we have

Example 4.6 Let X be a Hausdorff space, consider the following statements:

1) X contains no copies of the real line.
2) X is an arbitrary subset of the real line.
3) When $H = \mathcal{I}$ or $H = \mathbb{R}$, *the cancellation law*: $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds for all $A, B \subseteq X$.
4) X contains no copies of the real plane $\mathbb{R}^2$.

Then: (1)⇒(3) by Theorem 1.2. (2)⇒(3) by Propositions 4.1 and 4.3. Finally, from example 1.4, it follows that (3)⇒(4). Obviously: (3) may not imply either (1) or (2). However, we conjecture that (4) implies (3), when X is a Peano continuum.

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