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FOLIATED BUNDLES AND METRIC RIGIDITY *

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Abstract

We obtain a Hermitian metric rigidity theorem for foliated vector bundles other than the leafwise tangent bundle, allowing us to develop results not considered in [29]. In particular, we obtain as a consequence a vanishing theorem for the leafwise cohomology of a foliation, which has as a corollary a partial vanishing result of certain holomorphic 1-forms on suitable compact Kaehler manifolds, and a rigidity property for holomorphic equivalences of foliations by irreducible bounded symmetric domains with complex dimension ≥ 2.

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1 Motivation

The symmetric spaces of non-compact type and their quotients have been a source of a great deal of study. One of the first important steps into the understanding of the relations between the geometry and topology of compact quotients of such spaces was given by Mostow's strong rigidity theorem.

Theorem (Mostow[25])**.** *Let X and Y be compact quotients of symmetric spaces of non-compact type with the same fundamental group. If X*

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has no closed two dimensional local factor, then X and Y are isometric up to normalizing constants.

Later on, Mostow's theory was extended with Margulis' work by considering the fundamental group of a quotient of a symmetric space as a discrete subgroup of a semisimple Lie group, and then proceeding to study the extension problem for homomorphisms defined on such subgroups. A particular case of the main result is:

Theorem (Margulis' superrigidity [20]). *Let G be a connected centerless semisimple Lie group without compact factors rank ≥ 2, Γ an irreducible lattice of G and H a non-compact simple Lie group. If $\pi : \Gamma \to H$ is a homomorphism with Zariski dense image, then π has a continuous extension to a homomorphism $G \to H$.*

Since any connected centerless semisimple Lie group is the connected component of the group of isometries of a symmetric space, it follows from the basic properties of symmetric spaces that the extension of π induces a totally geodesic map of the symmetric spaces associated to G and H. Of course, the homomorphism π can be considered as coming from a representation of the fundamental group of a quotient of the symmetric space associated to G, so that the above consequence of Margulis' superrigidity can be expressed in purely geometric terms.

Theorem (Margulis' geometric superrigidity). *Let $\tilde{X}$, $\tilde{Y}$ be symmetric spaces of non-compact type. Assume that $\tilde{Y}$ is irreducible and $\mathrm{rank}(\tilde{X}) \geq 2$, and let $X = \tilde{X}/\Gamma$ be an irreducible finite volume quotient. If $\rho : \Gamma \to \mathrm{Isom}(\tilde{Y})$ is a representation with Zariski dense image, then there is a ρ-equivariant totally geodesic map $\tilde{X} \to \tilde{Y}$.*

From this point, there have been several routes followed in an attempt to generalize or extend the superrigid properties of semisimple Lie groups without compact factors and the symmetric spaces associated to them. In the context of Lie group actions, Zimmer [36] extended Margulis' superrigidity to a cocycle superrigidity which has proved useful in the study of actions of semisimple Lie groups without compact factors.

On the other hand, from a geometric point of view there are several natural generalizations to both Margulis' superrigidity and Mostow's strong rigidity. First it was proved that the strong rigidity of symmetric spaces is a phenomenon that takes place in the broader context of manifolds with non-positive curvature.

Theorem (Gromov[3], Ballmann-Eberlein[2]). *Let X be an irreducible finite volume quotient of a symmetric space of rank ≥ 2. Let Y be a complete Riemannian manifold of bounded non-positive sectional curvature and finite volume. If $\pi_1(X) \cong \pi_1(Y)$, then X and Y are isometric up to normalizing constants.*

Given this result, it was natural to attempt to extend the above theorem on Margulis' geometric superrigidity to the context of representations of lattices of semisimple Lie groups into groups of isometries of non-positively curved Riemannian manifolds. A natural result one can seek is the following general problem of the geometric Archimedean superrigidity for irreducible symmetric spaces:

Problem . *Let Ω be an irreducible symmetric space of non-compact type, Γ the fundamental group of a finite volume quotient of Ω and $\rho : \Gamma \to \mathrm{Isom}(\tilde{N})$ a homomorphism into the group of isometries of a complete simply connected Riemannian manifold of non-positive sectional curvature. Find restrictions on the choices of ρ and $\tilde{N}$ that imply the existence of an isometric totally geodesic embedding $\Omega \hookrightarrow N$.*

This problem has been satisfactorily solved by using existence theorems of ρ-equivariant harmonic maps and Bochner formulae techniques. A remarkable fact about geometric superrigidity is that it has allowed to prove superrigidity of lattices in the groups $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$ (cf. [5]), a case which was not considered in Margulis' work.

Assuming the existence of ρ-equivariant harmonic maps, Mok-Siu-Yeung [23] have provided a comprehensive study of the Bochner techniques used in geometric superrigidity. Their work, together with an existence theorem of ρ-equivariant harmonic maps in the case of $\tilde{N}$ symmetric due to Corlette [4], provides a geometric proof of the Archimedean case of Margulis' superrigidity. This existence result of Corlette has been extended by Labourie to include more general manifolds $\tilde{N}$:

Theorem (Labourie[18]). *Let M be a compact Riemannian manifold, $\tilde{N}$ a simply connected complete Riemannian manifold with non-positive sectional curvature and $\rho : \pi_1(M) \to \mathrm{Isom}(\tilde{N})$ a homomorphism. If $\mathrm{Im}(\rho)$ is geometrically reductive, then there is a ρ-equivariant harmonic map $f : \tilde{M} \to \tilde{N}$.*

We refer to Labourie [18] for a complete explanation of the term geometrically reductive. In this paper the existence is attained by using the heat equation so that, starting with any ρ-equivariant map, we move through a homotopy to get a harmonic map. And it is a homotopy invariant of an arbitrary ρ-equivariant map $f : \Omega \to \tilde{N}$, i.e. the image of ρ, and its non-triviality, i.e. geometric reductivity, that allows to say that the existence of such map f implies the existence of a harmonic map. This in turn becomes a totally geodesic map (under certain hypothesis) by using the Bochner formula of Mok-Siu-Yeung. To illustrate this we state the following easy consequence of Labourie [18] and Mok-Siu-Yeung [23]:

Theorem. *Let M be a compact quotient of an irreducible symmetric space Ω of non-compact type which is either of rank ≥ 2 or a Cayley or quaternionic hyperbolic space. Let $\tilde{N}$ be a complete simply connected Riemannian manifold with non-positive sectional curvature in the case* $\mathrm{rank}(\Omega) \geq 2$ *and with non-positive complexified sectional curvature otherwise, and let $\rho : \pi_1(M) \to \mathrm{Isom}(\tilde{N})$ be a homomorphism with geometrically reductive image. If $f : \Omega \to \tilde{N}$ is a ρ-equivariant map, then f is homotopic to an isometric totally geodesic embedding.*

This result provides a solution for the following restatement of the geometric superrigidity problem, at least for suitable choices of Ω and $\tilde{N}$.

Problem (Geometric Archimedean superrigidity)**.** *Let M be a compact quotient of an irreducible symmetric space Ω of non-compact type with rank ≥ 2, $\tilde{N}$ a complete simply connected Riemannian manifold with non-positive sectional curvature, $\rho : \pi_1(M) \to \mathrm{Isom}(\tilde{N})$ a homomorphism and $f : \Omega \to \tilde{N}$ a ρ-equivariant smooth map. Find a homotopy invariant of f whose non-triviality implies the existence of an isometric totally geodesic embedding $\Omega \hookrightarrow \tilde{N}$.*

The last theorem shows that $\rho(\pi_1(M))$ is one such homotopy invariant if we call it non-trivial whenever it is geometrically reductive. On the other hand, Margulis' geometric superrigidity uses the same homotopy invariant but in this case non-triviality holds when $\rho(\pi_1(M))$ is Zariski dense. In both cases a common hypothesis for the solution of the above geometric Archimedean superrigidity is the compactness of the manifolds involved or at least a bound on the volume and curvature. This remark suggests to consider the problem of geometric superrigidity in a broader context that includes manifolds with some kind of bounded geometry.

Compact manifolds with smooth foliations carrying smooth leafwise Riemannian metrics provide such setup. Even though a leaf in one such foliation is not in general a finite volume quotient of its universal cover, it has been possible to extend results given for finite volume manifolds to this kind of Riemannian manifolds. One of the many examples of this remark is the following:

Theorem (Strong rigidity for foliations, Zimmer[33]). *Let M and N be standard Borel spaces with measurable foliations carrying measurable leafwise smooth Riemannian metrics that arise from free ergodic irreducible actions of connected centerless semisimple Lie groups G and H, respectively. So that in particular, the leaves are locally symmetric. Assume that* $\operatorname{rank}(G) \geq 2$. *If M and N are transversally equivalent, then there is an isomorphism of measurable foliations which is (up to normalizing constants) an isometry on a.e. leaf.*

We recall that two foliations M and N as above are transversally equivalent if they have transversals T and T', respectively, for which there is a Borel isomorphism $T \to T'$ that preserves the equivalence relation induced by the foliation as well as the measure class. For a complete discussion of the relevance of these concepts we refer to [33] and [34]. Here we want to point out that it is the equivalence relation of the transversal that determines how the leaves are packed into the foliation, so it plays a role similar to that of the fundamental group of a Riemannian manifold in providing a way to build a finite volume or compact object out of a non-compact geometric model.

Given this strong rigidity for foliations and the above statement for the geometric Archimedean superrigidity it is natural to consider the following:

Problem A (Geometric superrigidity for foliations). *Let M and N be compact smooth manifolds with smooth leafwise Riemannian metrics and let $f : M \to N$ be a smooth leaf preserving map. Assume each leaf of M is isometrically covered by a fixed irreducible symmetric space Ω of non-compact type, and that each leaf of N has non-positive sectional curvature. Find a homotopy invariant for f whose non-triviality implies the existence of an isometric totally geodesic immersion of Ω into some or most of the leaves of N.*

A variation of this problem can be obtained by replacing N above with a Riemannian manifold.

Problem B . *Let M be compact smooth manifold with a smooth leafwise Riemannian metric, N a compact Riemannian manifold with non-positive sectional curvature and $f : M \to N$ a smooth map. Assume* *that each leaf of M is isometrically covered by a fixed irreducible symmetric space Ω of non-compact type. Find a homotopy invariant for f whose non-triviality implies the existence of an isometric totally geodesic immersion of $\Omega \to N$.*

Solutions to Problems A and B were developed in [30]. On the other hand, it is known that the geometric rigidity for symmetric spaces established in one of the above Theorems due to Gromov, Ballmann and Eberlein is even stronger for Hermitian symmetric spaces (symmetric spaces which have a Kaehler structure compatible with its Riemannian metric). The following result due to Mok is an example of such claim:

Theorem (Mok[21])**.** *Let X be a compact quotient of an irreducible Hermitian symmetric space of rank ≥ 2. Let h be a Kaehler metric on X with nonpositive holomorphic bisectional curvature. Then, h is a constant multiple of the canonical metric.*

We refer to [21] for further details and applications to holomorphic maps on bounded symmetric domains. Following the idea of introducing foliations as compact models for symmetric spaces, it was proved in [29] a result similar to the above which provides a foliated Hermitian metric rigidity:

Theorem (Quiroga [29])**.** *Let M be a compact manifold with a smooth foliation carrying a finite invariant transverse measure μ, a leafwise holomorphic structure and a smooth leafwise Kaehler metric g. Assume that each leaf is uniformized by a fixed irreducible bounded symmetric domain Ω with rank ≥ 2. If h is a smooth leafwise Hermitian metric with non-positive curvature in the sense of Griffiths, then g and h are homothetic on μ-a.e. leaf of M.*

The main contribution of this paper within this setup is to show that the above foliated Hermitian metric rigidity extends to a foliated Hermitian metric rigidity for holomorphic bundles other than the tangent bundle.

2 Introduction

This paper contains a reasonably complete presentation of results which establish Hermitian metric rigidity for suitable foliated vector bundles. However, most of the proofs are omitted and will appear elsewhere. Our chief result is the following:

Main Theorem. *Let M be a compact manifold with a smooth foliation carrying a finite invariant transverse measure μ, a leafwise holomorphic structure and a leafwise Kaehler structure g. Assume that with g each leaf is uniformized by a fixed irreducible bounded symmetric domain Ω. Let (V, h_0) be a Hermitian vector bundle over M which is holomorphic, irreducible, locally homogenous and properly seminegative in the sense of Griffiths along the leaves. If h is a Hermitian metric on V with seminegative curvature in the sense of Griffiths along the leaves, then h is homothetic to h_0 on μ-a.e. leaf.*

In the next section we will explain the notions considered in this statement, here we remark that the conditions on (V, h_0) are satisfied by the leafwise holomorphic tangent bundle $T^{1,0}_{\mathcal{F}}M$ (the bundle of complex vectors of type $(1,0)$ tangent to the leaves) for $\mathrm{rank}(\Omega) \geq 2$ and for the bundle $S^{n+1}T_{\mathcal{F}} \otimes K_{\mathcal{F}}$ in the case $\Omega = B^n$, the unit ball, where $S^{n+1}T_{\mathcal{F}}$ is the $n+1$ symmetric power of the leafwise holomorphic tangent bundle and $K_{\mathcal{F}}$ is the leafwise canonical bundle.

From the Main Theorem we obtain the following rigidity property for holomorphic equivalences of foliations, which is in the spirit of the results by Zimmer and others found in [27], [33], [34] and [35].

Theorem A. *Let M be a compact manifold and $\mathcal{F}$, $\mathcal{F}'$ be smooth leafwise holomorphic foliations with the same leaf dimension and carrying leafwise Kaehler metrics g and g', respectively, such that the leaves of $\mathcal{F}$ are uniformized by a fixed irreducible bounded symmetric domain Ω. Assume that $\mathcal{F}$ has a finite invariant transverse measure μ and that either one of the following is satisfied:*

1. $\mathrm{rank}(\Omega) \geq 2$ *and the leaves of $\mathcal{F}'$ have non-positive holomorphic bisectional curvature, or*
2. *both the leaves of $\mathcal{F}$ and $\mathcal{F}'$ are uniformized by B^n.*

If $f\colon M \to M$ is a smooth map that defines a leafwise holomorphic equivalence from $\mathcal{F}$ onto $\mathcal{F}'$, then f is up to a constant an isometry on μ-a.e. leaf of $\mathcal{F}$.

Case (1) in this result can be seen as a consequence of Theorem 4.6 in [29], however case (2) cannot be obtained from [29].

We also deduce the following vanishing theorems for leafwise holomorphic sections of suitable vector bundles:

Theorem B. *Let M and (V, h_0) be as in the Main Theorem, and let F be a smooth leafwise holomorphic complex vector bundle over M carrying a Hermitian metric for which F is locally flat when restricted to the leaves. Then any smooth leafwise holomorphic section σ of $V^* \otimes F$ vanishes on μ-a.e. leaf of M.*

As an immediate consequence of this we obtain the following:

Theorem 6.1. *Let M be as in the Main Theorem with Ω an irreducible bounded symmetric domain of rank ≥ 2 and let F be a smooth leafwise holomorphic complex vector bundle over M carrying a Hermitian metric for which F is locally flat when restricted to the leaves. Then any smooth leafwise holomorphic F-valued 1-form vanishes on μ-a.e. leaf of M. In particular, if μ is positive on open sets, then the leafwise cohomology group $H^{1,0}_{\mathcal{F}}(M) = 0$.*

As a corollary to this result we obtain the following vanishing criterion for smooth leafwise harmonic 1-forms:

Theorem 6.2. *Let M be as in the Main Theorem with Ω an irreducible bounded symmetric domain of rank ≥ 2. Then any smooth leafwise harmonic 1-form vanishes on μ-a.e. leaf; in particular, any such form is identically zero for μ positive on open sets.*

This result should be compared to the techniques found in Zimmer [39] where the infinitesimal rigidity proven there has as an essential step a vanishing criterion for measurable leafwise smooth harmonic 1-forms defined over some particular suspensions. Our result requires global smoothness but it applies to more general foliations and hence can be considered as a step in developing the tools required to improve the results of [39] that are needed to show property (V) (see [39]), for lattices in the group G of isometries of Ω as above, which leads to strong rigid properties for actions of G on principal bundles.

Theorem 6.1 also provides a partial vanishing for holomorphic 1-forms on suitable Kaehler manifolds.

Theorem 6.4. *Let (X, g) be a compact Kaehler manifold such that $(\tilde{X}, g)$ has as a de Rham factor an irreducible bounded symmetric domain Ω of rank ≥ 2. If F is a locally flat Hermitian holomorphic vector bundle over X, then any holomorphic F-valued 1-form ω vanishes on vectors whose lifts to $\tilde{X}$ are tangent to Ω. In particular, this partial vanishing is satisfied for any holomorphic 1-form on X.*

3 Foliated vector bundles

In this section we recollect some notions about foliations and complex vector bundles that will be used throughout this work.

For a smooth foliation in a manifold M a transverse section is a closed subset of M that intersects every leaf in a non-empty countable set. Such transverse sections always exist and a finite measure on one of them which is invariant under the holonomy of the foliation is called a finite invariant transverse measure. We refer to [24] for a complete discussion of this concept.

In this work we systematically consider what is known as a leafwise structure on a foliation. The basic object that is used for such construction is the leafwise tangent bundle of a manifold M carrying a smooth foliation $\mathcal{F}$, denoted by $T_{\mathcal{F}}M$, which is defined as the subbundle of the tangent bundle consisting of all vectors tangent to the leaves. Given $T_{\mathcal{F}}M$ we say that $\mathcal{F}$ is leafwise holomorphic if there is a tensor J of type $(1,1)$ on $T_{\mathcal{F}}M$ such that $J^2 = -id$ and if the almost complex structure defined this way on each leaf is integrable. Given this, we can consider the holomorphic tangent bundle denoted by $T_{\mathcal{F}}^{1,0}M$ and spanned on every leaf by the vector fields $\partial/\partial z^i$ coming from holomorphic coordinates. A leafwise Kaehler structure g on $\mathcal{F}$ is a Riemannian metric on $T_{\mathcal{F}}M$ whose restriction to every leaf is a Kaehler metric for the given holomorphic structure J. This allows us to consider all basic notions from Riemannian geometry along the leaves of a foliation when these structures are given. In particular, we have the leafwise exterior differential $d_{\mathcal{F}}$, the leafwise Levi-Civita connection $D_{\mathcal{F}}$ and the complex operator $\partial_{\mathcal{F}}$; all these operators and some other geometric objects will be considered only leafwise and hence we will denote them with the usual symbology dropping the $\mathcal{F}$ as subindex.

In the rest of this work M will denote a compact manifold with a smooth foliation $\mathcal{F}$ carrying a leafwise holomorphic structure. Let V be a complex vector bundle over M which we assume to be smooth, we will say that V is leafwise holomorphic if its restriction to each leaf is a holomorphic vector bundle. On such V a Hermitian metric h always determines a unique leafwise Hermitian connection D, which is defined only along the leaves since the holomorphic structure is given in general only leafwise. The operator D^2, known as the curvature of h, defines a leafwise $V \otimes \bar{V}$-valued 2-form of type $(1,1)$ denoted by $\Theta(h)$ or symply by Θ. Θ in turn induces at every point $x \in M$ a Hermitian bilinear map:

$$E_x \otimes T_x^{1,0}(X) \times E_x \otimes T_x^{1,0}(X) \to \mathbb{C}$$
$$(e \otimes \alpha, e' \otimes \alpha') \mapsto \Theta(e, \bar{e}', \alpha, \bar{\alpha}') = \Theta_{e\bar{e}'\alpha\bar{\alpha}'}$$

And we say that (V, h) is seminegative in the sense of Griffiths when this map is non-positive, and we call it properly seminegative in the sense of Griffiths (at some point) when it is seminegative but not negative definite.

Now let Ω be an irreducible bounded symmetric domain with G the identity component of its group of isometries and K the corresponding isotropy at a point $o \in \Omega$. We will denote with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ the Cartan decomposition of the Lie algebra of G, where $\mathfrak{k}$ denotes the Lie algebra of K and $\mathfrak{m}$ is identified with $T_o\Omega$. Under this last identification the decomposition $T_o^{\mathbb{C}}\Omega = T_o^{1,0}\Omega \oplus T_o^{0,1}\Omega$ induces a corresponding decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$. Also recall that the Lie algebra $\mathfrak{k}$ of K decomposes into the sum of its semisimple part $\mathfrak{k}_s$ and its 1-dimensional center $\mathfrak{z}$.

Assume that V_0 is a finite dimensional complex vector space acted upon by K, we will say that such representation is irreducible if it is so for the representation induced on $\mathfrak{k}_s$. When this is the case V_0 has a unique (up to scalar multiplication) K-invariant Hermitian inner product. Such a representation of K provides a holomorphic vector bundle over Ω defined by $V = (G \times V_0)/K$, for the diagonal action of K on $G \times V_0$. Any such bundle is known as a homogeneous vector bundle over Ω, and it is further called irreducible homogeneous when the representation of K is irreducible; in this last case, there is a unique (up to scalar multiplication) G-invariant Hermitian metric over V which will be denoted by h_0. If X is a (Kaehler) quotient of Ω, then V induces a holomorphic vector bundle V' on X which is known as a locally homogeneous vector bundle and further called irreducible locally homogeneous when V is irreducible.

Now assume that M carries a leafwise Kaehler structure g such that each leaf is uniformized by a fixed irreducible bounded symmetric domain Ω. A leafwise holomorphic complex vector bundle V over M is called leafwise irreducible locally homogeneous when its restriction to each leaf is an irreducible locally homogeneous vector bundle associated to a fixed representation of K for every leaf. With this and similar notions we sometimes replace the term leafwise by saying that a property holds along the leaves.

Example 1. Let M, $\mathcal{F}$ and g be as above with Ω an irreducible bounded symmetric domain of rank ≥ 2. Then the leafwise holomorphic tangent bundle $T_{\mathcal{F}}^{1,0}M$ is irreducible, locally homogeneous and properly seminegative in the sense of Griffiths along the leaves. The same holds true for (most) wedge powers of $T_{\mathcal{F}}^{1,0}M$; this last claim can be obtained from the well known formulas for curvature of wedge products and by writting down the action of K on a fiber of the bundles considered to prove irreducibility (cf. [9]).

Example 2. For M, $\mathcal{F}$, g as above and $\Omega = B^n$, the open unit ball in $\mathbb{C}^n$, the bundle $T_{\mathcal{F}}^{1,0}M$ is leafwise irreducible locally homogeneous but not properly seminegative. However, it is a simple matter to show that $S^{n+1}T_{\mathcal{F}} \otimes K_{\mathcal{F}}$ is leafwise irreducible, locally homogeneous and properly seminegative, where $S^{n+1}T_{\mathcal{F}}$ is the $n+1$ symmetric power of the leafwise holomorphic tangent bundle and $K_{\mathcal{F}}$ is the leafwise canonical bundle, (cf. [21]). This allows to obtain certain rigid properties for the rank 1 case which was not considered in [29].

If we are given two smooth foliations $\mathcal{F}$ and $\mathcal{F}'$ on our compact manifold M, a smooth leafwise holomorphic equivalence between them is defined by a diffeomorphism $f\colon M \to M$ which is leaf preserving and holomorphic along the leaves.

Finally the leafwise Dolbeaut cohomology $H_{\mathcal{F}}^{p,q}(M)$, for a manifold M with a leafwise holomorphic structure, is defined as in the case of a complex manifold using leafwise differential forms and the operator $\bar{\partial}_{\mathcal{F}}$. In particular, $H_{\mathcal{F}}^{1,0}(M)$ is the space of leafwise $(1,0)$ forms which are holomorphic (along the leaves). We refer to [24] for more details on related constructions.

4 Sketch of the proof of the Main Theorem

From now on Ω will denote a fixed irreducible bounded symmetric domain and G, K, $\mathfrak{k}$, $\mathfrak{k}_s$ and so on will denote the objects previously discussed.

In order to prove the Main Theorem one takes advantage of the zeros for the curvature of a properly seminegative vector bundle. Since we are dealing with symmetric spaces it should not be surprising to be able to characterize the values for which the curvature vanishes in terms of representations of the Lie algebra $\mathfrak{k}$. On the other hand, it was remarked in [21] and [29] that highest root vectors in $\mathfrak{m}^+$ associated to certain Cartan subalgebras of $\mathfrak{g}^{\mathbb{C}}$ realize the minimum for the holomorphic sectional curvature (these were called characteristic vectors in the cited works). Furthermore, such vectors realize what is known as the null dimension of Ω (cf. [29]) allowing us to optimize the use of the zeros for the curvature.

Now consider as before an irreducible homogeneous holomorphic vector bundle V over Ω carrying a G-invariant Hermitian metric. In this setup we have the following generalization of the notion considered in [21] and [29]:

Definition 4.1. A unitary vector v_0 in a seminegative irreducible homogeneous holomorphic vector bundle V over Ω is called characteristic if together with some $\xi_0 \in T^{1,0}\Omega$ (at the same base point) it realizes the minimum of the map $(v, \xi) \mapsto \Theta_{v\bar{v}\xi\bar{\xi}}$

The following result from [21] shows that for this extended setup we can also describe the above distinguished vectors in terms of the Lie algebra theory associated to Ω.

Proposition 4.2. *The minimum (maximum) of the values of the map $(v, \xi) \mapsto \Theta_{v\bar{v}\xi\bar{\xi}}$ is attained at a pair (v_0, ξ_0) of unitary vectors if and only if we can choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$ for $\mathfrak{g}$ and some ordering for the corresponding root space decomposition such that ξ_0 is a highest root vector and v_0 is a highest (resp. lowest) weight vector for the representation of $\mathfrak{k}^{\mathbb{C}}$ that defines the bundle (V, h_0).*

Back to our foliated setup, let M be a compact manifold with a smooth leafwise holomorphic foliation $\mathcal{F}$ carrying a leafwise Kaehler structure g that turns each leaf into a quotient of Ω, and let (V, h_0) be a leafwise seminegative irreducible locally homogeneous vector bundle over M as described before. By projectivizing the fibers of V we obtain a smooth leafwise holomorphic fiber bundle $\pi\colon \mathbf{P}(V) \to M$. Within this bundle we distinguish the set of all characteristic vectors with the following:

Definition 4.3. The characteristic bundle of V is defined by:

$$\mathcal{M}(V) = \{[v] \in \mathbf{P}(V) \mid v \text{ is a characteristic vector of } V\}$$

and we will denote with $p\colon \mathcal{M}(V) \to M$ the restriction of the projection $\pi\colon \mathbf{P}(V) \to M$.

The following lemma can be easily proved using the homogeneity of the symmetric space Ω that covers the leaves of M (cf. [21]) together with the local triviality of the foliation $\mathcal{F}$.

Lemma 4.4. *$\mathcal{M}(V)$ is a compact manifold with a smooth leafwise holomorphic foliation carrying a finite invariant transverse measure $\mu_{\mathcal{M}}$ such that $p\colon \mathcal{M}(V) \to M$ is a leaf and measure preserving map that defines a smooth leafwise holomorphic fiber bundle.*

As in [29] we consider the tautological line bundle associated to V and defined by:

$$L(V) = \{(v, [u]) \in V \times \mathbf{P}(V) \mid [v] = [u] \text{ in } \mathbf{P}(V)\}$$

Using the arguments of [29] we obtain the following results:

Lemma 4.5. *The canonical map $L(V) \to \mathbf{P}(V)$ defines a smooth leafwise holomorphic line bundle whose restriction to each leaf is precisely the tautological line bundle for the restriction of V to the corresponding leaf. In particular, any Hermitian metric h on V defines a Hermitian metric $\hat{h}$ on $L(V)$.*

Proposition 4.6. *For M and (V, h_0) as before there is a leafwise cohomology class $c_1(L(V)) \in H^2_{\mathcal{F}}(\mathbf{P}(V); \mathbb{C})$ such that:*

$$c_1(L(V)) = \left[\frac{\sqrt{-1}}{2\pi}\Theta\right]$$

where Θ is the curvature of any leafwise connection for the smooth leafwise holomorphic line bundle $L(V) \to \mathbf{P}(V)$. In particular, if ρ is a Hermitian metric on $L(V)$, then the first Chern form of $(L(V), \rho)$ locally defined by:

$$c_1(L(V), \rho) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\rho_i$$

is a representative of $c_1(L(V))$, where $\rho_i = \rho(e_i, e_i)$ and e_i is a smooth leafwise holomorphic non-vanishing local section of $L(V)$.

As in [29] we conclude from this that, when (V, h_0) is seminegative in the sense of Griffiths, $\theta = -c_1(L(V), \hat{h}_0) + \pi^*(\omega_0)$ defines a leafwise Kaehler structure on $\mathbf{P}(V)$ where ω_0 is the leafwise Kaehler form on M coming from the metric g which is leafwise uniformized by Ω.

Finally, the principal tool in the proof of the Main Theorem is the following vanishing theorem which generalizes the corresponding results found in [21] and [29]. We remark that the Lemma 4.8 below constitutes a key ingredient in the proof.

Proposition 4.7. *Let M be a manifold with a smooth leafwise holomorphic foliation $\mathcal{F}$ carrying a finite invariant transverse measure and* *a leafwise Kaehler structure g which is leafwise uniformized by a fixed irreducible bounded symmetric domain Ω, and let (V, h_0) be a leafwise irreducible locally homogeneous holomorphic vector bundle which is seminegative in the sense of Griffiths. If h is a Hermitian metric over V, then:*

$$\int_{\mathcal{M}(V)} (-c_1(L(V), \hat{h}))^\alpha \wedge \theta^\beta d\mu_{\mathcal{M}} = 0$$

where $\alpha = \operatorname{rank}(c_1(L(V), \hat{h}_0)) + 1$ and β is such that $\alpha + \beta$ equals the complex dimension of the leaves of $\mathcal{M}(V)$. In particular, if h is seminegative in the sense of Griffiths, then:

$$c_1(L(V), \hat{h})^\alpha = 0$$

on $\mu_{\mathcal{M}}$-a.e. leaf of $\mathcal{M}(V)$.

Lemma 4.8. *For any point o in M the vector space $V_o \otimes T_o^{1,0}\mathcal{L}$ is spanned by the set:*

$$W = \{v \otimes \zeta \mid [v] \in \mathcal{M}(V), \Theta(h_0)_{v\bar{v}\zeta\bar{\zeta}} = 0\}$$

where $\mathcal{L}$ is the leaf through o.

With the above results the proof of the Main Theorem, which we ommit in this work, can be completed.

5 Rigidity for holomorphic equivalences of foliations

An important consequence of the Main Theorem is the following:

Theorem A. *Let M be a compact manifold and $\mathcal{F}$, $\mathcal{F}'$ be smooth leafwise holomorphic foliations with the same leaf dimension and carrying leafwise Kaehler metrics g and g', respectively, such that the leaves of $\mathcal{F}$ are uniformized by a fixed irreducible bounded symmetric domain Ω. Assume that $\mathcal{F}$ has a finite invariant transverse measure μ and that either one of the following is satisfied:*

1. $\operatorname{rank}(\Omega) \geq 2$ *and the leaves of $\mathcal{F}'$ have non-positive holomorphic bisectional curvature, or*
2. *both the leaves of $\mathcal{F}$ and $\mathcal{F}'$ are uniformized by B^n.*

If $f\colon M \to M$ is a smooth map that defines a leafwise holomorphic equivalence from $\mathcal{F}$ onto $\mathcal{F}'$, then f is up to a constant an isometry on μ-a.e. leaf of $\mathcal{F}$.

6 Vanishing of leafwise cohomology groups

The main theorem in this section is the following:

Theorem B. *Let M and (V, h_0) be as in the Main Theorem, and let F be a smooth leafwise holomorphic complex vector bundle over M carrying a Hermitian metric for which F is locally flat when restricted to the leaves. Then any smooth leafwise holomorphic section σ of $V^* \otimes F$ vanishes on μ-a.e. leaf of M.*

A consequence of the above is the following:

Theorem 6.1. *Let M be as in the Main Theorem with Ω an irreducible bounded symmetric domain of rank ≥ 2 and let F be a smooth leafwise holomorphic complex vector bundle over M carrying a Hermitian metric for which F is locally flat when restricted to the leaves. Then any smooth leafwise holomorphic F-valued 1-form vanishes on μ-a.e. leaf of M. In particular, if μ is positive on open sets, then the leafwise cohomology group $H^{1,0}_{\mathcal{F}}(M) = 0$.*

As an application we have:

Theorem 6.2. *Let M be as in the Main Theorem with Ω an irreducible bounded symmetric domain of rank ≥ 2. Then any smooth leafwise harmonic 1-form vanishes on μ-a.e. leaf; in particular, any such form is identically zero for μ positive on open sets.*

For our last application we recall the following remark from [29]:

Remark 6.3. Let (X, g) be a compact Kaehler manifold and assume that its universal cover decomposes isometrically $(\tilde{X}, g) \cong \Omega \times Y$ where Ω is a non-Euclidean de Rham factor. Such decomposition defines a foliation with leaves of the form $\Omega \times \{y_0\}$ and it carries an invariant transverse measure given by the volume of the factor Y. This foliation with invariant transverse measure does not in general induces a structure of the same sort on the manifold X; however it is a simple matter to show that there is finite cover $X' \to X$ so that the foliation on $\tilde{X}$ descends to X'. Moreover, being X' compact one can show that the invariant transverse measure on $\tilde{X}$ defines a finite invariant transverse measure over X'. Also observe that both the foliations on $\tilde{X}$ and X' are leafwise holomorphic and carry a leafwise Kaehler structure defined by the Kaehler metric on Ω.

Theorem 6.4. *Let (X, g) be a compact Kaehler manifold such that $(\tilde{X}, g)$ has as a de Rham factor an irreducible bounded symmetric domain Ω of rank ≥ 2. If F is a locally flat Hermitian holomorphic vector bundle over X, then any holomorphic F-valued 1-form ω vanishes on vectors whose lifts to $\tilde{X}$ are tangent to Ω. In particular, this partial vanishing is satisfied for any holomorphic 1-form on X.*

7 Further developments

The first section provides an introduction to the main ideas from the study of the rigid properties for symmetric spaces. In particular, it should allow the reader to understand the goals for this area of the geometry. However, a couple of problems to consider are briefly discussed.

We can obtain the same conclusion of the Main Theorem under weaker regularity assumptions on the metrics involved, as long as we can guarantee that the forms involved are suitable to use Stokes' theorem. This should prove to be useful in applications to measurable setups as in the work of Zimmer.

In [22] the single leaf case of the Main Theorem is used, together with a Bochner formula, to study representations of fundamental groups of compact quotients of Ω into the isometry group of a Riemannian manifold N with suitable non-positive curvature. The results in [22] have been improved with the use of a stronger Bochner formula introduced in [23]. An interesting problem to consider is to build the kind of results developed in [22] and [23] by using a foliated version of the Bochner formula introduced in [23]. This approach requires to establish the existence of leafwise harmonic maps. As noted from the results in [14] the work that has to be done is not straightforward, since leafwise harmonic maps are not easy to get. This is due to the lack of convergence of the leafwise heat flow, so that a way to control its behavior has to be used. This problem has been considered in [29] where a kind of geometric superrigidity for foliations is obtained together with applications to Riemannian manifolds. Within such setup, an interesting problem to consider is to develop a suitable Hodge theory for foliations that allows to use the leafwise vanishing theorems we consider in this work to obtain leafwise cohomology vanishing for certain foliations; such techniques have been used by Zimmer to obtain rigid properties for actions of semisimple Lie groups.

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