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ON THE ROLE OF GROUPS IN TOPOLOGY AND GEOMETRY *

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Abstract

In this article we develop some relationships between fundamental concepts in geometry and topology and structures in (mostly) the theory of finite groups. In particular, we stress the way in which classical results in group theory relate to topology and have given rise to new developments in homotopy theory. We describe work in progress motivated by these relationships both in low dimensional topology and in homotopy theory.

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1 Preliminaries on the classifying space construction

The n-dimensional simplex σ^n is one of the most fundamental objects in mathematics. Recall its definition:

$$\sigma^n \;=\; \{(t_1, \ldots, t_n) \mid 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1\},$$

with faces

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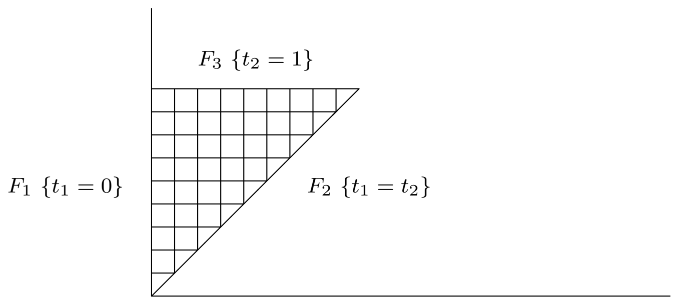
$$\begin{array}{rcl} F_1 & = & \sigma^n \cap \{t_1 = 0\} \\ F_2 & = & \sigma^n \cap \{t_1 = t_2\} \\ \vdots & & \ddots \\ F_n & = & \sigma^n \cap \{t_{n-1} = t_n\} \\ F_{n+1} & = & \sigma^n \cap \{t_n = 1\}. \end{array}$$



The simplex σ^2

Let $\mathcal{O}$ be a collection of objects, and $\mathcal{C}$ be a collection of "maps" between them with a *partially defined* associative multiplication on the maps. That is to say, there is a pairing on a subset of the elements of $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$ so that if fg and gh are both defined, then $(fg)h$ and $f(gh)$ are defined and equal. For example, in the set of complex manifolds, $\mathcal{C}$ could be the collection of holomorphic maps from one to another. (Note that the composition of two holomorphic maps is again holomorphic.)

Definition 1.1 *Let $(f_1, \ldots, f_r) \in \mathcal{C}^r$ be an ordered r-tuple of elements of $\mathcal{C}$ which has the property that $f_i f_{i+1} \in \mathcal{C}$, $1 \leq i \leq r-1$. Then we call $(f_1, \ldots, f_r)$ an r-composable set, ($\mathcal{C}$ is thus the collection of all 1-composable sets).*

Consider the space $B_{\mathcal{C}}$ constructed out of the collection of objects and all r-composable sets for all $r \geq 1$ by taking the disjoint union

$$\coprod_{\mathcal{O}} e \times \sigma^o \ \sqcup \ \coprod_r \coprod_H \sigma^r \times H$$

as H runs over all r-composable sets and making the identifications in the relation set $\mathcal{R}$ that couple face identifications in the simplex with *composition in the r-composable set $(f_1, \ldots, f_r)$:*

$$
\begin{aligned}
(\vec{t}, (f_1, \ldots, f_r)) &\sim ((t_2, \ldots, t_r), (f_2, \ldots, f_r)) \\
&\qquad\qquad \textit{if } \ \vec{t} \in F_1(\sigma^r). \\
&\sim ((t_1, \ldots, \widehat{t_i}, \ldots, t_r), (f_1, \ldots, \widehat{f_i}, f_i f_{i+1}, \ldots, f_r)) \\
&\qquad\qquad \textit{if } \ \vec{t} \in F_{i+1}(\sigma^r), \quad 1 \leq i \leq r-1. \\
&\sim ((t_1, \ldots, t_{r-1}), (f_1, \ldots, f_{r-1})) \\
&\qquad\qquad \textit{if } \ \vec{t} \in F_{r+1}(\sigma^r),
\end{aligned}
$$

and finally that couple r-composable sets $(f_1, \ldots, f_r)$ with the object e_r on which f_r is defined, and e_0 which is the image of f_1.

This space is called the **classifying space of $\mathcal{C}$** and is a basic construction in modern mathematics. There are also reduced constructions where we take special account of identity maps, but this need not concern us here.

$\mathcal{C}$ is always assumed to have a topology, and if none is given then the discrete topology is assumed. Similarly for the objects. Additionally, the simplex σ^r has the induced topology from that of the r-cube I^r. The quotient space above is then given the *compactly generated* quotient topology.

Remark 1.2 A special case is the situation where $\mathcal{C}$ is simply a space with a continuous associative multiplication: for example a topological group (or a discrete group) and $\mathcal{O}$ consists of a single point regarded as both the domain and range of each element of $\mathcal{C}$. In case Y is a connected and based CW-complex and $X = \Omega Y$ is the loop space of Y with loop sum as its product, then $B_X \simeq Y$ where the symbol $\simeq$ denotes homotopy equivalence. Thus, up to homotopy type, all cellular spaces occur this way.

Examples 1.3 (a) If G is a discrete group and there is again only one object, then $B_G = K(G, 1)$, the Eilenberg-Mac Lane space. B_G is characterized by the property that $\pi_1(B_G) = G$, $\pi_i(B_G) = 0$ for $i \geq 2$, and, consequently, B_G is universal for principal G-covers, [5].

(b) For the continuous group S^1, $B_{S^1} \simeq \mathbb{CP}^\infty$.

(c) Suppose that $\mathcal{C} = \mathcal{D} \times \mathcal{E}$, then if $\mathcal{C}$ and $\mathcal{D}$ are compact CW-complexes with unit elements 1, the reduced classifying spaces satisfy

$$\bar{B}_{\mathcal{C}} \;=\; \bar{B}_{\mathcal{D}} \times \bar{B}_{\mathcal{E}}$$

in the compactly generated topology.

(d) Let $X = SP^\infty(Y)$ be the infinite symmetric product of the based space Y, given as $\lim_n(SP^n(Y))$ where the inclusion

$$SP^i(Y) \hookrightarrow SP^{i+1}(Y),$$

is given by $\langle y_1, \ldots, y_i\rangle \mapsto \langle y_1, \ldots, y_i, *\rangle$. Then $B_X = SP^\infty(\Sigma Y)$, where σY is the *reduced* suspension of Y, [4].

(e) Suppose X is a finite simplicial complex with simplices $\sigma^r_{i_r}$. Let the objects $\mathcal{O}$ be the simplices of X, and we define $\mathcal{C}$ as follows: f_{i_r,i_s} is defined if and only if $\sigma^r_{i_r}$ is a face of $\sigma^s_{i_s}$. The composition $f_{i_r,i_s} f_{i_t,i_l}$ is defined and equals f_{i_r,i_l} if and only if $\sigma^s_{i_s}$ is a face of $\sigma^t_{i_t}$. Then $B_{\mathcal{C}}$ is exactly the barycentric decomposition of X and is thus homeomorphic to X.

2 A key connection between finite group theory and homotopy theory

For a finite group G with G' perfect, the "plus"-construction B^+_G is obtained by adding 2-cells and 3-cells to B_G that "abelianize" its fundamental group without changing the integral homology of any of its abelian covers. Thus $\pi_1(B^+_G) = G/G'$ and $H_*(B_G, \mathbb{A}) \to H_*(B^+_G; \mathbb{A})$ is an isomorphism for all twisted coefficients $\mathbb{A}$ for which the twisting factors through $\mathbb{Z}(G/G')$.

A fundamental connection between finite group theory and homotopy theory is based on the homotopy equivalence

$$Q(S^0) = \lim_{n \mapsto \infty}(\Omega^n S^n) \simeq \Omega B\left(\coprod B_{\mathcal{S}_n}\right) \simeq \mathbb{Z} \times B^+_{\mathcal{S}_\infty}, \tag{2.1}$$

where the associative multiplication in $\coprod B_{\mathcal{S}_n}$ is induced from the usual inclusions $\mathcal{S}_n \times \mathcal{S}_m \subset \mathcal{S}_{n+m}$, and where $\Omega^n S^n$ is the space of based self-maps $S^n \to S^n$. The space $Q(S^0)$ plays a key role in studying the structure of mapping spaces. For example, note the following result.

Remark 2.1 By definition, the (unstable) homotopy groups of $Q(S^0)$ are the stable homotopy groups of spheres: $\pi_i(Q(S^0)) = \pi_i^s(S^0) = \lim_{n\mapsto\infty} \pi_{n+i}(S^n)$.

Quillen used the equivalence in (2.1) to construct a splitting

$$(B^+_{GL_\infty(\mathbb{F}_p)})_q \times Coker(J)_q \simeq Q(S^0)_q$$

for q appropriately related to p and p odd. Here the splitting is obtained from the usual inclusion of $\mathcal{S}_n$ into $GL_n(\mathbb{F}_p)$ and the inclusion $GL_n(\mathbb{F}_p) \subset \mathcal{S}_{p^n-1}$ via the action on the non-zero vectors of $\mathbb{F}_p^n$.

There is a second product defined on $Q(S^0)$ since the elements of this space represent limits of (based) maps $f_m \colon S^m \longrightarrow S^m$, given by composing the maps. It ties in to the construction above via group homomorphisms

$$\mathcal{S}_n \times \mathcal{S}_m \longrightarrow \mathcal{S}_{nm}$$

given as the compositions

$$\mathcal{S}_n \times \mathcal{S}_m \xrightarrow{\Delta^n \times 1} \mathcal{S}_n \wr \mathcal{S}_m \hookrightarrow \mathcal{S}_{nm}$$

where the inclusion of the wreath-product $\mathcal{S}_n \wr \mathcal{S}_m \hookrightarrow \mathcal{S}_{nm}$ is the usual one. This follows from 3.2(c) and is the basis for the main results of [6].

3 Internal structure of the symmetric groups

In view of the discussion above, the internal structure of the symmetric groups should have corresponding relations to structure in homotopy theory. Particularly important are the *maximal subgroups* of $\mathcal{S}_n$. These are given by the O'Nan-Scott theorem:

Theorem 3.1 *A maximal subgroup of $\mathcal{S}_n$ has one of the following six forms:*

(i) $\mathcal{S}_n \times \mathcal{S}_m \subset \mathcal{S}_{n+m}$.

(ii) $\mathcal{S}_m \wr \mathcal{S}_k \subset \mathcal{S}_{mk}$.

(iii) $\mathcal{S}_m \wr \mathcal{S}_n \subset \mathcal{S}_{m^n}$.

(iv) $Aff_n(p) \subset \mathcal{S}_{p^n}$.

(v) $G \subset \mathcal{S}_n$ *where* $T \triangleleft G \xrightarrow{\psi} L$ *with* T *a non-abelian simple group but not* $\mathcal{A}_n$, $L \subset Aut(T)$, *and the associated action* ψ *is primitive.*

(vi) $(G \wr \mathcal{S}_n) \cdot Out(G) \subset \mathcal{S}_{|G|^{n-1}}$ *where* G *is a simple group, not* $\mathcal{A}_n$.

Remarks 3.2 (a) The loop sum in (2.1) corresponds to the subgroups $\mathcal{S}_n \times \mathcal{S}_m \subset \mathcal{S}_{n+m}$ in Theorem 3.1. (See the discussion in [6], particularly pps. 241–243.)

(b) The "transfer" operation $Q(Q(X)) \to Q(X)$ given by

$$\Omega^n\Sigma^n(\Omega^n\Sigma^n X) \mapsto \Omega^n\Sigma^n X,$$

using the intermediate map $eval\colon \Sigma^n\Omega^n Y \to Y$, corresponds here to the maximal type (ii) subgroups $\mathcal{S}_k \wr \mathcal{S}_n \subset \mathcal{S}_{nk}$ through the following chain of equivalences.

$$\begin{aligned} Q(Q(S^0)) \quad &\sim \quad \coprod E_{\mathcal{S}_k} \times_{\mathcal{S}_k} Q(S^0)^k / \sim \\ &= \quad \coprod_{k,n} E_{\mathcal{S}_k} \times (E_{\mathcal{S}_n})^k \times_{\mathcal{S}_k \wr \mathcal{S}_n} *^{nk} / \sim' . \end{aligned}$$

(c) In the special case of S^n, we can regard $\Omega^n S^n$ as the set of based maps

$$\mathrm{Map}_*(S^n, S^n),$$

and, as such, we can *iterate* mappings $(f, g) \mapsto f \circ g$. This gives a second multiplication on $\Omega^n S^n$, and after passing to limits, on $Q(S^n)$. The associated pairing here is the one

$$\mathcal{S}_k \times \mathcal{S}_n \xrightarrow{\Delta^n \times 1} \mathcal{S}_k \wr \mathcal{S}_n \hookrightarrow \mathcal{S}_{nk}$$

defined by regarding nk points as n rows of k points. Then act by $\mathcal{S}_k$ diagonally on the rows and by $\mathcal{S}_n$ diagonally on the columns. The type (iii) maximal subgroups correspond to the analog of the construction of the Dyer-Lashof operations (equivalently, the transfer map of 3.2 (b)) for the composition product. (Again, see [6], pps. 241–243.)

In the late 1960's this led to the determination of the cohomology of the classifying space of $Q(S^0)_{\pm 1}$ under composition product, [6]. Using this, Brumfiel, Madsen, and Milgram [2] analyzed the structure of the surgery classifying spaces B_{TOP} and B_{PL} and extended the work of Sullivan on the fibers of the maps $G/TOP \to B_{TOP}$ and $G/PL \to B_{PL}$. Among the consequences were the determination of the PL-bordism ring as well as all possible integral characteristic classes obtained as polynomials in the Pontrjagin classes for topological manifolds.

4 The significance of the remaining classes of groups

In this section we discuss on the meaning of the remaining classes of groups in the O'Nan-Scott theorem. See also [7]. In one of the great works of mathematics in this century the finite simple groups were classified [3]. The result is that they are the

1. central quotients of the (simply connected) finite groups of Lie type, among them the groups $PSL_n(q)$, $PU_n(q)$, the exceptional groups $G_2(q)$, $F_4(q)$, $E_8(q)$, etc.,
2. the alternating groups,
3. the sporadic groups: 26 extraordinary finite groups the first five of which were discovered at the end of the last century by Mathieu: M_{11}, M_{12}, M_{22}, M_{23}, and M_{24}.

The first interesting case is $i\colon M_{12} \subset \mathcal{A}_{12} \subset \mathcal{S}_{12}$. M_{12} and $G_2(q)$ with $q \equiv 3, 5 \bmod (8)$ share a common 2-Sylow subgroup, and

$$H^*(G_2(q); \mathbb{F}_2)$$

has a 14-dimensional aspect which we can see in its Poincaré series

$$\begin{aligned} P_{G_2(q)}(t) &= \frac{(1+t^3)(1+t^5)(1+t^6)}{(1-t^4)(1-t^6)(1-t^7)} \\ &= \frac{1+t^3+t^5+t^6+t^8+t^9+t^{11}+t^{14}}{(1-t^4)(1-t^6)(1-t^7)}. \end{aligned} \tag{4.1}$$

The cohomology ring $H^*(M_{12}, \mathbb{F}_2)$ and consequently the Poincaré series for $H^*(M_{12}, \mathbb{F}_2)$ is determined in [1], and we have

$$\begin{aligned} &(1-t^4)(1-t^6)(1-t^7)P_{M_{12}}(t) = \\ &\quad = 1+t^2+3t^3+t^4+3t^5+4t^6+2t^7+4t^8+3t^9 \\ &\qquad +t^{10}+3t^{11}+t^{12}+t^{14}. \end{aligned} \tag{4.2}$$

An algebra over a field F is Cohen-Macaulay if it is free and finitely generated over a polynomial subalgebra. We have the following result.

Theorem 4.1 ([1], [8]) *$H^*(M_{12};\mathbb{F}_2)$ is Cohen-Macaulay over*

$$H^*(B_{G_2};\mathbb{F}_2) = \mathbb{F}_2[d_4, d_6, d_7]$$

with the numerator of the Poincaré series having dimension 14. Moreover, there is a composition

$$S^{14} \to Q(B_{G_2}) \xrightarrow{tr'} Q(B_{G_2(3)}) \to Q(B_{M_{12}}) \xrightarrow{Q(B_i)} Q(B_{\mathcal{S}_{12}}) \to Q(S^0),$$

where the image of S^{14} in $H^*(M_{12})$ represents the 14 dimensional fiber generator. Here tr' is the composition of the transfer to $(\mathbb{Z}/2)^3 \times \mathcal{S}_4$ followed by inclusion into M_{12}, and in $\pi_{14}(Q(S^0)) = \pi^s_{14}(S^0)$ the S^{14} represents an element of Kervaire invariant one.

Remark 4.2 The 2-Sylow subgroup here has the simple structure given as the semi-direct product

$$(\mathbb{Z}/4)^2 \colon (\mathbb{Z}/2)^2 = \langle a, b\rangle \colon \langle k, t\rangle,$$

with $a^t = a^{-1}$, $b^t = b^{-1}$, $a^k = b$ describing the action of the generators of the $(\mathbb{Z}/2)^2$ on $(\mathbb{Z}/4)^2$. In $G_2(q)$, $q \equiv 3, 5 \bmod (8)$ the 2-primary part of the maximal torus is also a copy of $(\mathbb{Z}/4)^2$. The Weyl group is $(\mathbb{Z}/2) \times \mathcal{S}_3$, so the normalizer of the 2-part of the torus is

$$(\mathbb{Z}/4)^2 \colon (\mathbb{Z}/2) \times \mathcal{S}_3.$$

This extension is also part of M_{12} and in both groups it contains the 2-Sylow subgroup.

In the case above we had a simple group which was related to a group of Lie type by having a map to a classifying space with plus construction of the homotopy type of the classifying space of G_2. Moreover, there was a degree one map of the resulting fiber onto a space having the homotopy type of a Lie group.

The fact that a Lie group is parallelizable results in a stable spherical class, due to the following well known result:

Lemma 4.3 *Let M be any stably parallelizable closed, compact n-dimensional manifold without boundary, then some large suspension of M, $\Sigma^k M \simeq S^{k+n} \vee W$ where W has dimension $< k+n$.*

What is striking here is the fact that the usual inclusions of $G_2(q) \subset \mathcal{S}_n$ do not map this class to the class σ^2, but the natural map associated to M_{12} does. And that is, perhaps, the point. One is looking for more or less *natural* constructions which give rise to exotic objects. Clearly aspects of this process can be generalized. But it should be emphasized that currently we do not know what the results of this type of construction are in full generality.

Remark 4.4 Let us emphasize again what seems to be happening here. The structures we are interested in, such as elements in the image of the stable Hurewicz homomorphism in the homology of a simple group, already occur in the Sylow subgroups. However, it is (hopefully) the relation of the simple group with the symmetric groups which gives rise to the natural maps that carry these elements to stable homotopy theory in interesting ways.

As further support for this it has been shown in my analysis of the sporadic group M_{23}, [9], that $H_i(M_{23}; \mathbb{F}_2) = 0$ for $i \leq 5$, $H_6(M_{23}, \mathbb{F}_2) = \mathbb{F}_2$, and the inclusion

$$M_{23} \subsetneq \mathcal{S}_{23} \hookrightarrow \mathcal{S}_\infty$$

on passing to classifying spaces and plus constructions takes the first non-zero mod (2) homotopy class of $B^+_{M_{23}}$ to the 6-dimensional Kervaire class. But there may be much more going on here than just this though what is happening in higher dimensions is not entirely clear. The Sylow 2-subgroup of M_{23} is the same as that of the sporadic group McL and that of the group of Lie type $PSU_4(\mathbb{F}_3)$ so we would, at least, expect that the properties of $PSU_4(\mathbb{F}_3)$ would lead to further homotopy theoretic conclusions.

Motivated partly by the remarks above, it seems that the most promising groups to study in order to generalize what is happening here are the groups $PSU_n(q)$ with q odd ($q = 3$ comes to mind in view of the close connection between $PSU_4(3)$ and 4 of the remaining sporadic groups).

In ongoing work I have been studying these groups with Kristin Umland at the University of New Mexico. So far we have about finished with understanding the structure of the elementary 2-groups in $PSU_n(q)$ and $PSL_n(q)$. This involves an analysis of the modular representations and automorphism groups of the various hyperelementary groups, inductively given as the central products

$$H * Q_8, \qquad H * D_8, \qquad \text{and} \qquad H * \mathbb{Z}/4,$$

if $H \neq \mathbb{Z}/4$, and with the single relation induces from the isomorphism

$$Q_8 * Q_8 \cong D_8 * D_8,$$

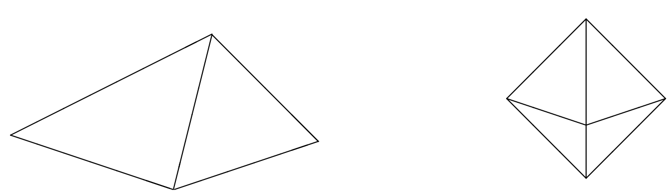
so the first groups are

$$\mathbb{Z}/4,\ Q_8,\ D_8,\ D_8 * Q_8,\ D_8 * D_8,\ D_8 * \mathbb{Z}/4,\ Q_8 * D_8 * D_8,\ D_8 * D_8 * D_8.$$

It is interesting to note that these groups have consistently played critical roles in investigations in homotopy theory and manifold theory in general. For example, they were critical in studying the classical classifying space $B_{Spin(n)}$ and also play critical roles in the structure of vector fields on spheres.

5 Some interactions of finite group theory with low dimensional topology

The tetrahedral group, octahedral group, and icosahedral groups all are contained in $SO(3)$ as are the cyclic groups $\mathbb{Z}/n$ and the dihedral groups D_{2n}.



The tetrahedron and octahedron

The two fold covering $Sp(1) \cong S^3 \to SO(3)$ lifts these groups to their binary analogues, the binary tetrahedral group, binary octahedral group, the binary icosahedral group, the cyclic groups again, and the generalized quaternion groups Q_{4n}, which consequently act freely on S^3 and there are three dimensional manifolds with these as their fundamental groups: the quotients of S^3 by the actions above.

The classification theory for 3-dimensional closed manifolds naturally bifurcates into the two cases: $\pi_1(M^3)$ is infinite and $\pi_1(M^3)$ is finite – the case of Thurston's elliptic conjecture. It is believed that we are very near a complete understanding of the M^3 with infinite π_1. However, we really know very little about the situation for π_1 finite. For example, we do not even know if the groups above are the *only possible finite fundamental groups*. However, due to work of Milnor and R. Lee we know that the only remaining possibilities are groups of the form

$$\mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c \colon {}_T Q_8, \ a, \ b, \ c \text{ coprime},$$

where the twisting is via the action of the first generator of Q_8 as inversion on $\mathbb{Z}/a$, the second generator on $\mathbb{Z}/b$ and both generators on $\mathbb{Z}/c$.

In work of mine from almost 20 years ago on actions on S^{11} it was shown that if two or more of a, b, c are not equal to 1, then there are stringent constraints on the possible elements which can occur, the first possibility being

$$\mathbb{Z}/17 \times \mathbb{Z}/113 \colon {}_T Q_8.$$

Recently, using a result of Rubenstein, I strengthened the results above to prove

Theorem 5.1 *If one of the groups*

$$\mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c \colon {}_T Q_8$$

is the fundamental group of a closed, compact 3*-manifold* M^3*, with* $(2, a, b, c)$ *mutually coprime and two or more of* a*,* b*,* c *greater than* 1*, then the universal cover of* M^3 *is a counterexample to the Poincaré conjecture.*

In a recent paper by I. Hambleton and R. Lee it was claimed that the known groups are the only possibilities. However, it seems there is a serious error in the details. Nonetheless, the philosophy there seems convincing, and the theorem above lends support to the conjecture that none of these groups can be the fundamental group of a closed, compact 3-manifold. So it is now expected that this will be the case.

However, even in the case of cyclic groups there remains the question of how many homeomorphically distinct lens spaces there are with fundamental group $\mathbb{Z}/n$. In the case where $n = 2^s 3^t$ work of Rubenstein shows that there is only one. However, in the remaining cases nothing is known.

The first invariant which can be non-trivial for n involving a prime $p \geq 5$ is a Reidemeister type torsion class. Let $R = \mathbb{Z}(\mathbb{Z}/n)$ be the integral group ring. Then, setting $G = Aut_R(R^n)$, $n \geq 2$ this invariant takes its values in $H_1(B_G;\mathbb{Z})$, factored out by a small subgroup. In fact, the result of 5.1 is a direct consequence of the following result.

Theorem 5.2 *Suppose that an exotic group $\mathbb{Z}/a \times \mathbb{Z}/b \times \mathbb{Z}/c\colon {}_TQ_8$ is the fundamental group of a closed compact 3-manifold M^3, then there is a lens space with fundamental group $\mathbb{Z}/(2ab)$, $\mathbb{Z}/(2ac)$ or $\mathbb{Z}/(2bc)$ with a non-trivial torsion invariant.*

It has been widely conjectured for a long time that there are no exotic lens spaces. However, the first critical case is clearly $\mathbb{Z}/5$ where the possible torsion invariant would be

$$T + T^{-1} - 1.$$

In studying whether this class can be the torsion of an exotic L_5 one runs into presentations of groups of a highly symmetric form. The first of them corresponding to the simplest presentation associated to the unit above is

$$F(x_1,\dots,x_5)/(x_1x_2^{-1}x_3, x_2x_3^{-1}x_4, x_3x_4^{-1}x_5, x_4x_5^{-1}x_1, x_5x_1^{-1}x_2).$$

There will also be more complex relators given by the relators above multiplied by products of commutators of the generators but always with the properties that

1. there is an action of $\mathbb{Z}/n$ cyclically permuting the generators,
2. the relation set is generated by a single relator together with other $(n-1)$ distinct images under the action of $\mathbb{Z}/n$.

What we seem to need now are techniques which will decide when such groups are non-trivial. I conjecture that they are always non-trivial in the cases of interest.

This would not quite solve the problem of classification of lens spaces but it would, at least, show that there are no exotic fundamental groups.

6 Current Developments

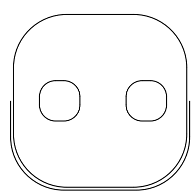
In some sense what we have been discussing to this point reflects advances on ideas which have been around for some time. But recently new ideas from younger workers notably Ulrike Tillmann at Oxford and V. Voevodsky at Harvard and Northwestern have given us new tools which promise to revolutionize the subject.

One of the basic groups which connects analysis, number theory and topology is $SL_n(\mathbb{Z})$ the group of $n \times n$ integer matrices of determinant one. And beyond that the next level of groups which one expects to play a critical role in developments in the next 50 years are the mapping class groups consisting of the groups of automorphisms of the fundamental groups of closed 2-manifolds, $\Gamma^g_{n,m}$ is the set of isotopy classes of diffeomorphisms of the Riemann surface of genus g with n punctures and m boundary components. The classifying spaces for these groups have been a major objective of a lot of mathematics for perhaps the last 50 years.

In both cases there are stabilization maps. For $SL_n(\mathbb{Z})$ this takes the form of the standard inclusion homomorphisms

$$SL_n(\mathbb{Z}) \hookrightarrow SL_{n+1}(\mathbb{Z}), \qquad A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

For the group $\Gamma^g_{n,m}$ one glues in a pair of pants over a piece of one of the boundary components.



(the heavy part of the arc is glued to a corresponding arc in the first boundary component) or, in the same way attach a torus with an open disk $\dot{D}^2$ removed.

So it makes sense to pass to limits, obtaining Γ^∞_n and $SL(\mathbb{Z})$.

Work of Kontsevich based on previous work of Penner some years ago showed the existence of polynomial rings on 2-dimensional generators, $\mathbb{Z}[b_1, \ldots, b_n]$, in $H^*(B\Gamma_n)$. Also, work of E. Miller, Morita, and

Mumford showed that $H^*(B\Gamma_0)$ contains a polynomial algebra on even generators

$$\mathbb{Z}[b_2, b_4, \ldots, b_{2n}, \ldots].$$

But beyond this, little was known about these spaces until about 2 years ago. Then Ulrike Tillmann proved that $B_{\Gamma_n^\infty}$ is an infinite loop space which severely limits the structure it can have. In particular she proved that it has a splitting component $Q(S^0)$. It is now suspected that the remaining components are something like $B_O \times \Omega^2 B_0$ producted with a number of $\mathbb{CP}^\infty$'s. Also, based on her work she, Maginnis and I showed the existence of unstable torsion classes in $H_*(B_0^5)$ and it is believed that there are unstable torsion free classes as well. But the unstable problem seems very complicated.

Recently, Voevodsky introduced new techniques and proved a conjecture of Milnor. From this Charles Weibel [10] obtained the structure of $K_*(\mathbb{Z})$, the homotopy groups of $B^+_{SL(\mathbb{Z})}$ as follows:

$$\begin{aligned}
K_{8n}(\mathbb{Z}) &= (odd) \text{ for } n \geq 1\\
K_{8n+1}(\mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z}/2 \oplus (odd) \text{ for } n \geq 1\\
K_{8n+2} &= \mathbb{Z}/2 \oplus (odd)\\
K_{8n+3}(\mathbb{Z}) &= \mathbb{Z}/16 \oplus (odd)\\
K_{8n+4}(\mathbb{Z}) &= (odd)\\
K_{8n+5} &= \mathbb{Z} \oplus (odd)\\
K_{8n+6}(\mathbb{Z}) &= (odd)\\
K_{8n+7}(\mathbb{Z}) &= (\mathbb{Z}/w_i) \oplus (odd),\ i = 4(n+1)
\end{aligned}$$

where w_i is the largest power of 2 dividing $4i$.

In fact Luke Hodgkin gives the following description: Let $JK(\mathbb{Z})$ be defined as the fiber of the composite map

$$c(\psi^3 - 1)\colon BO \xrightarrow{\psi^3-1} BSpin \xrightarrow{c} BSU.$$

Then localized at the prime two $JK(\mathbb{Z})$ is $BGL(\mathbb{Z})^+$.

Thus we are looking at what promises to be a very exciting time in this area. I eagerly await the subsequent developments.

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