Morfismos, Vol. 2, No. 1, 1998, pp. 23–37

UNIQUE FACTORIZATION IN CARTESIAN PRODUCTS *

EDUARDO SANTILLÁN [1]

Abstract

In this paper we partially answer the following question. "If the topological space H can be decomposed as the Cartesian product $H = A \times X$, when is this factorization unique?"

1991 Mathematics Subject Classification: 54B10, 54A99, 54B15.
Keywords and phrases: Unique factorization law, (pathwise) connected, totally (pathwise) disconnected, cut point, Peano continuum, closed-Peano-connectedness.

1 Introduction.

Suppose that topological space H can be decomposed in the Cartesian product $H = A \times X$. When is this factorization unique? That is, given four topological spaces A, B, X and Y such that $A \times X \cong B \times Y$, when is A homeomorphic to B and X homeomorphic to Y? This is a very old question in topology without a complete answer so far, and results which give conditions for the answer to be *positive* are known as *unique factorization laws*. The main purpose of this paper is to present several unique factorization laws all of which are deduced from the following key lemma (proved in section 2).

Lemma 1.1 *Let A, B, X and Y be four topological spaces which satisfy that $A \times X \cong B \times Y$ under a homeomorphism g. Assume that for every*

*This paper is a portion of the author's doctoral thesis written at the Departamento de Matemáticas, CINVESTAV-IPN.

[1] Currently at York University.

$a \in A$, there exists a point $b \in B$ such that $g(\{a\} \times X) \subseteq \{b\} \times Y$; and that for every $b \in B$, there exists a point $a \in A$ such that $\{b\} \times Y \subseteq g(\{a\} \times X)$; then $A \cong B$ and $X \cong Y$.

A unique factorization law is trivially obtained if a least two of four spaces A, B, X and Y are degenerate (that is, with cardinality at most one). Thus, by space we mean *non-degenerate* topological space.

There are several non-trivial unique factorization laws. For instance a space H is said to be *prime* if it cannot be decomposed into a Cartesian product of two or more spaces which are not singletons nor homeomorphic to H. Thus "no polyhedron of any dimension can have more than one decomposition into a product of (non-degenerate) prime sets of dimension not exceeding 1" [2] and "no closed n-dimensional manifold (that is, a space locally homeomorphic to Euclidean n-dimensional space) can have more than one decomposition into a product of (non-degenerate) prime sets of dimension not exceeding 2" [3]. There are also many products with no unique factorization.

Example 1.2 No two of the unit intervals (0,1), [0,1) and $\mathcal{I} = [0, 1]$ in the real line $\mathbb{R}$ (endowed with the standard topology) are homeomorphic; however, it is easy to prove that the following three products are homeomorphic: $(0, 1) \times [0, 1), [0, 1) \times [0, 1)$ and $\mathcal{I} \times [0, 1)$. Another example was found by R. H. Fox in 1947 (see[5]). He constructed two closed non-homeomorphic subsets A_F and B_F of $\mathbb{R}^2$ such that $A_F \times \mathcal{I} \cong B_F \times \mathcal{I}$.

On the other hand, a recent factorization law has been given by E. Behrends and J. Pelant in [1]: Let H be a compact, connected, Hausdorff space such that the only continuous mappings from H to H are the constant and the identity; then for arbitrary compact Hausdorff spaces A and B, the law $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds. Finally, the proof of Lemma 1.1 is presented in section 2, together with definitions and background material. The applications of Lemma 1.1 are presented in sections 3 and 5 as unique factorization laws (we consider Theorems 3.1, 3.2, 5.3 and 5.5 to be the main results of this paper). And section 4 is dedicated to prove some preliminary results which are used in the proof of Theorems 5.3 and 5.5.

2 Basic Definitions and Lemmas.

We begin this section by showing Lemma 1.1.

Proof of Lemma 1.1: Given $a \in A$, there exist elements $b \in B$ and $c \in A$ such that $\{a\} \times X \subseteq g^{-1}(\{b\} \times Y) \subseteq \{c\} \times X$. This implies $a = c$ (X is non-empty), so that $g(\{a\} \times X) = \{b\} \times Y$ and $X \cong Y$.

Now define $R : A \to B$ by $R = \psi \circ g \circ \varphi^{-1}$, where $\varphi : A \times X \to A$ and $\psi : B \times Y \to B$ are the projections. Given $a \in A$, from the preceding paragraph, we obtain $R(a) = \psi \circ g \circ \varphi^{-1}(a) = \psi \circ g(\{a\} \times X) = \psi(\{b\} \times Y) = b$. We can also show that for every $b \in B$, there exists a unique $a \in A$ such that $R(a) = b$. Therefore $R : A \to B$ is a one-to-one and onto function. On the other hand, φ and ψ are continuous and open mappings, so R is also a continuous and open mapping, so that $R : A \to B$ is a homeomorphism. ∎

Recall now some definitions taken from [4], [6], [7] and [8].

Definition 2.1 *Let H and X be arbitrary spaces, then:*

1. *When X is connected, $x \in X$ is a cut point if $X - \{x\}$ is not connected;*
2. *H is totally disconnected if every connected set contained in H is degenerate;*
3. *X is pathwise connected if for all $x, z \in X$ there exists a path (a continuous mapping f from $\mathcal{I}$ into X) such that $f(0) = x$ and $f(1) = z$;*
4. *H is totally pathwise disconnected if the only paths in H are the constant;*
5. *The pair of points $x, z \in X$ is arcwise connected if $x = z$ or if there exists an arc $\Upsilon \subseteq X$ with end points x and z;*
6. *A continuum is a connected and compact space;*
7. *A Peano continuum is a metrizable and locally connected continuum;*
8. *H is punctform if every continuum subset $D \subseteq H$ is a singleton;*
9. *H is said to be KC if every compact subset $D \subseteq H$ is closed;*
10. *The function $f : H \to X$ is said to be a perfect mapping if it is a closed, continuous and surjective mapping such that for every $x \in X, f^{-1}(x)$ is compact;*

11. *The subset* $\mathcal{E} \subseteq 2^X$ *is said a 2-cover of* X *if for all* $x, z \in X$ *there exists* $D \in \mathcal{E}$ *such that* $x, z \in D$.

We also need three basic lemmas.

Lemma 2.2 *Given two connected spaces* X *and* Y*, the product* $X \times Y$ *has no cut points.*

Proof: Recalling that X and Y are non-degenerate, the lemma follows from the arguments in Theorem 46.II.11 of [7 page 137].

Lemma 2.3 *Let* $f : X \to Y$ *be a continuous, closed and onto mapping and let* X *be a Peano continuum. Then* Y *is also a Peano continuum.*

Proof: Since X is T_1 and f is closed, Y is T_1. Thus, given $y \in Y$, the set $f^{-1}(y) \subseteq X$ is closed and; therefore, compact. Hence f is a perfect mapping and the result now follows from the Hahn-Mazurkiewicz theorem and the Hanai-Morita-Stone theorem; see [6]. ∎

Thus Peano continua are preserved under perfect mappings. On the other hand, in Lemma 2.3, since f is a continuous mapping, it sends closed (compact) subsets of X to compact subsets of Y; thus, it is possible to change the condition that f be closed for the condition that Y be a KC space. This yields the following lemma.

Lemma 2.4 *Let* $f : X \to Y$ *be a continuous mapping. If* X *is a Peano continuum and* Y *is a* KC *space, then* $f(X) \subseteq Y$ *is a closed Peano continuum.*

Finally, we introduce a new definition inspired by Theorem 3.2 and Example 3.5.

Definition 2.5 *Two points* x *and* z *are closed-Peano-connected in the space* X *(written* $x \overset{p}{\sim} z$*) if* $x = z$ *or if there exists a closed Peano continuum* $D \subseteq X$ *such that* $x, z \in D$*. Moreover,* X *is closed-Peano-connected if it has a closed 2-cover consisting of Peano continua (that is, every pair of points in* X *is closed-Peano connected).*

In section 4 we analyze properties of closed-Peano-connectedness. For example, we show that $\overset{p}{\sim}$ is an equivalence relation. Therefore, the following is well defined.

Definition 2.6 *The* closed-Peano-components *of the space* X *are the equivalence classes of* $\overset{p}{\sim}$ *in* X.

3 Unique Factorization Laws.

There exist many spaces which satisfy the hypotheses of Lemma 1.1. In this section we present two main unique factorization laws deduced from this lemma and their applications.

Theorem 3.1 *Let A, B, X and Y be spaces such that $A \times X \cong B \times Y$ and let Ξ be a topological property that satisfies:*

1. *Ξ is invariant under continuous mappings,*
2. *X and Y have property Ξ,*
3. *every non-empty subset of A with property Ξ is a singleton,*
4. *B satisfy the same hypothesis than A.*

Then $A \cong B$ and $X \cong Y$.

In Theorem 3.1, Ξ may stand for *connectedness*, *$\sigma-$ connectedness, compactness plus connectedness, pathwise connectedness*, etc.

Proof: Let g be a homeomorphism from $A \times X$ onto $B \times Y$ and let $\psi : B \times Y \to B$ be the projection (ψ is a continuous mapping). Given $a \in A$, since X is non-empty and has property $\Xi, (\psi \circ g)(\{a\} \times X) \subseteq B$ is also non-empty and has property Ξ. Hence, by condition 4 in 3.1, $(\psi \circ g)(\{a\} \times X) = \{b\}$ with $b \in B$, so $g(\{a\} \times X) \subseteq \psi^{-1}(b) = \{b\} \times Y$. We can likewise prove that for every $b \in B$, there exists a point $a \in A$ such that $\{b\} \times Y \subseteq g(\{a\} \times X)$. The result follows then from Lemma 1.1. ∎

Theorem 3.2 *Let A, B, X and Y be spaces such that $A \times X \cong B \times Y$. Suppose that A and B are T_1 and there exists a topological property Θ which satisfies:*

1. *Θ is invariant under perfect mappings and implies compactness,*
2. *X and Y both have a closed 2-cover with property Θ,*
3. *Every non-empty closed subset of A with property Θ is a singleton,*
4. *B satisfy the same hypothesis than A.*

Then $A \cong B$ and $X \cong Y$.

In Theorem 3.2, a space H has the property Θ if:

i) H is compact, connected and Hausdorff;
ii) H is compact, pathwise connected and metrizable;
iii) H is compact, σ-connected and metrizable;
iv) H is a Peano continuum.

In order to prove Theorem 3.2 we need the following.

Lemma 3.3 *Let Θ be a topological property that satisfy condition 1 in Theorem 3.2, that is, invariant under perfect mappings and implies compactness. Let $\psi : B \times C \to B$ be the projection. If $D \subseteq B \times C$ is closed and has property Θ, then $\psi(D)$ is closed in B and also has property Θ.*

Proof: Let $\varphi : B \times C \to C$ be the projection; then $\varphi(D)$ is compact because D is compact. On the other hand, it is easy to check that the restriction $\psi \mid_{B\times\varphi(D)}$ is the projection mapping from $B \times \varphi(D)$ to its first factor B so that $\varphi \mid_{B\times\varphi(D)}$ is a continuous closed mapping (see [4] and [6]). Moreover, since $D \subseteq B\times\varphi(D)$ is closed, the restriction of ψ to D, $\psi \mid_D$: $D \to \psi(D) \subseteq B$ is also a continuous, closed, and onto mapping such that $\psi \mid {}^{-1}_{D}(b) = D \cap (\{b\} \times \varphi(D))$ is compact for every $b \in \psi(D)$. Hence ψ_D is a perfect mapping and the set $\psi(D) = \psi \mid_{B\times\varphi(D)} (D) = \varphi \mid_D (D)$ has the property Θ and is closed in B. ∎

Proof of Theorem 3.2: Let g be a homeomorphism from $A \times X$ onto $B \times Y$ and let $\psi : B \times Y \to B$ be the projection. Since A is T_1 given $a \in A$ the set $E = g(\{a\} \times X) \subseteq B \times Y$ is closed, non-empty and has a closed 2-cover with property Θ.

We claim that $\psi(E)$ is a singleton. Take $s \in E$ and let $b = \psi(s)$. Then we have for every $t \in E$ a closed subset $D \subseteq E$ with property Θ such that $s, t \in D$. Since D is also closed in $B \times Y$, we deduce, applying Lemma 3.3, that $\psi(D)$ is closed in B and has property Θ. Therefore, by condition 4, $\psi(D) = \psi(t) = b$, so that $\psi(E) = b$ and $E \subseteq \psi^{-1}(b) = \{b\} \times Y$. Thus for every $a \in A$ there exists a point $b \in B$ such that $g(\{a\} \times X) \subseteq \{b\} \times Y$; we can likewise prove that for every $b \in B$ there exists a point $a \in A$ such that $\{b\} \times Y \subseteq g(\{a\} \times X)$. The result then follows from Lemma 1.1. ∎

There are many applications of the previous theorems.

Example 3.4 Let X and Y be connected spaces, and let A, B be totally disconnected spaces. Then the law $A \times X \cong B \times Y \Leftrightarrow A \cong B$ and $X \cong Y$ holds.

We can change the pair of properties (connected, totally disconnected) by (continua, punctform) or (pathwise connected, totally pathwise disconnected), in Example 3.4, and deduce other unique factorization laws. Even so, we cannot apply properties such as *Hausdorff* or *metrizable* in Theorem 3.1, because they are not invariant under continuous mappings. In this case we should use Theorem 3.2. We present an example using Peano continua.

Example 3.5 Let X and Y be closed-Peano-connected spaces, and let A, B be T_1 spaces such that every closed Peano continuum contained in A (or B) is a singleton. Then the unique cancellation law $A \times X \cong B \times Y \Leftrightarrow A \cong B$ and $X \cong Y$ holds.

On the other hand, Example 3.4 requires that both A and B be totally disconnected. We may weaken this condition as follows.

Theorem 3.6 *The unique factorization law $A \times X \cong B \times Y \Leftrightarrow A \cong B$ and $X \cong Y$ holds when X and Y are connected spaces, X has a cut point, A is totally disconnected and B is an arbitrary space.*

Proof: Let g be a homeomorphism from $A \times X$ onto $B \times Y$. We claim that B is totally disconnected. Suppose B has a non-degenerate connected component $\mathcal{E}$. By Lemma 2.2, $\mathcal{E} \times Y$ is a connected component of $B \times Y$ without cut points. On the other hand, since A is totally disconnected, for each $a \in A$ the set $\{a\} \times X$ is a connected component of $A \times X$. Hence there exists a point $a \in A$ such that $g(\{a\} \times X) = \mathcal{E} \times Y$. But this is a contradiction because X has a cut point. Thus B is totally disconnected and the result follows from Example 3.4. ■

We can likewise change the pair of properties (*connected, totally disconnected*) for (*continua, punctform*) or (*pathwise connected, totally pathwise disconnected*) in the last theorem. Moreover, in section 5, we are going to relax the hypotheses of Example 3.5 to require that A is a T_1 space, every closed Peano continuum contained in A is a singleton and B is an arbitrary space. Thus, the last two sections of this paper are dedicated to improve Example 3.5.

We will argue as in Theorem 3.6, but we need first to show that there exist *closed-Peano-components* (being closed-Peano-connected an equivalence relation), and that closed-Peano-components are preserved under Cartesian products (that is, given a closed-Peano-connected space Y, a set $E \subseteq X$ is a closed-Peano-component of X if and only if $E \times Y$ is also a closed-Peano-component of $X \times Y$).

4 Closed-Peano-Connectedness.

In this chapter we analyze the main properties of the closed-Peano- connectedness. We begin by noting the following. Since two points in a non-degenerate Peano continuum can be joined by a closed arc and every arc is a Peano continuum, we deduce an alternative characterization.

Lemma 4.1 *The points x and z are closed-Peano-connected in the space X if $x = z$ or if there exists a closed arc $\Upsilon \subseteq X$ with end-points x and z.*

Warning: one should be careful when working with closed-Peano-connectedness. Indeed, taking the space X and a proper subset $E \subset X$, if a pair of points $x, z \in E$ are closed-Peano-connected in E, then x and z are arcwise connected in X, but they are not necessarily closed-Peano-connected in X (unless E is closed in X); that is, there exists a closed arc $\Upsilon \subseteq E$ with end points x and z, but this arc may not be closed in X. Hence we must always specify which is the base space where two points are closed-Peano-connected. On the other hand, taking a closed arc $\Upsilon \subseteq X$, since Υ is T_1, every singleton contained in Υ is closed in X. Thus, from Lemma 4.1 we deduce the following.

Lemma 4.2 *Let $x, z \in X$ be different points such that $x \overset{p}{\sim} z$. Then the sets $\{x\}, \{z\} \subseteq X$ are closed. Moreover, every closed-Peano-connected space is T_1.*

On the other hand, from Lemmas 2.4 and 4.1 we obtain the following sequence of implications (x and z are different points in the space X).

$$
\begin{array}{lcl}
X \text{ is a Peano Continuum} & & \\
\Downarrow & & \\
\left(\begin{array}{l} x \text{ and } y \text{ are} \\ \text{closed-Peano-} \\ \text{connected in } X \end{array}\right) & \begin{array}{c} \Longrightarrow \\ \\ \Leftarrow +KC \end{array} & \begin{array}{c} x \text{ and } y \text{ are arcwise connected} \\ \Downarrow \\ x \text{ and } y \text{ are pathwise connected.} \end{array}
\end{array}
$$

We present now the main properties of closed-Peano-connectedness.

Theorem 4.3 *The relation of being closed-Peano-connected is an equivalence relation.*

Proof: Clearly $\overset{p}{\sim}$ is reflexive and symmetric. Let a, b and c be different points in the space X such that $a \overset{p}{\sim} b$ and $b \overset{p}{\sim} c$. Thus there exist closed arcs $\Upsilon_1, \Upsilon_2 \subseteq X$ whose end-points are a, b and b, c respectively. Therefore, we can construct a continuous and surjective mapping $f : \mathcal{I} \to \Upsilon_1 \cup \Upsilon_2 \subseteq X$.

Now, let $D \subseteq \Upsilon_1 \cup \Upsilon_2$ be compact. Since $\Upsilon_1 \subseteq X$ is closed, $D \cap \Upsilon_1$ is compact; and since Υ_1 is KC, we get that $D \cap \Upsilon_1$ is closed in both Υ_1 and X. Similarly, $D \cap \Upsilon_2 \subseteq X$ is closed. Then $D \subseteq \Upsilon_1 \cup \Upsilon_2$ is closed as well, and so $(\Upsilon_1 \cup \Upsilon_2)$ is KC. Hence $\Upsilon_1 \cup \Upsilon_2 \subseteq X$ is a closed Peano continuum (apply Lemma 2.4) with $a, c \in \Upsilon_1 \cap \Upsilon_2$; and $a \overset{p}{\sim} c$. ■

Theorem 4.4 *Let $f : X \to H$ be a continuous, closed and onto mapping. If X is closed-Peano-connected, then so is H.*

Proof: Given $a, b \in H$, there exist points $s, t \in X$ and a closed arc $\Upsilon \subseteq X$ such that $f(s) = a, f(t) = b$ and $s, t, \in \Upsilon$. Then $f\,|_{\Upsilon} \colon \Upsilon \to H$ is a continuous and closed mapping, so $f(\Upsilon) = f\,|_{\Upsilon}\,(\Upsilon) \subseteq H$ is a closed Peano continuum (apply Lemma 2.3) such that $a, b \in f(\Upsilon)$. Thus, H is closed-Peano-connected. ■

Theorem 4.4. says that closed-Peano-connectedness is invariant under continuous closed mappings; moreover, closed-Peano-connectedness is also invariant under topological products, as the following theorem states.

Theorem 4.5 *$X \times Y$ is closed-Peano-connected if and only if X and Y are both closed-Peano-connected spaces.*

Proof: Necessity: Let $x, z \in X$ be two different points and let $\varphi : X \times Y \to X$ be the projection map. Take $s \in Y$. Then there exists a closed Peano continuum $D \subseteq X \times Y$ such that (x, s), $(z, s) \in D$. By Lemma 3.3, $\varphi(D) \subseteq X$ is a closed Peano continuum with $x, z \in \varphi(D)$; this means that $x \overset{p}{\sim} z$. Likewise, Y is closed-Peano-connected.

Sufficiency: Given $(x, s), (z, t) \in X \times Y$, there exists a closed arc $\Upsilon \subseteq X$ with end points x and z. But $X \times \{s\}$ is closed in $X \times Y$ because Y is a T_1 space (see Lemma 4.2), so that $\Upsilon \times \{s\}$ is closed in $X \times Y$ and $(x, s) \overset{p}{\sim} (z, s)$. Likewise, $(z, s) \overset{p}{\sim} (z, t)$. Finally, from Theorem 4.3, $(x, s) \overset{p}{\sim} (z, t)$. ∎

Another consequence of Theorem 4.3 is that we may consider the closed-Peano- components in Definition 2.6. Using Lemma 4.1, we may characterize these components as follows.

Lemma 4.6 *The closed-Peano-component $\mathcal{C}_x$ of a point $x \in X$ is the set composed by the point x and all points $z \in X$ such that there exists a closed arc $\Upsilon \subseteq X$ with end points x and z, that is, $\mathcal{C}_x = \{x\} \cup \{z \in X \mid$ there exists a closed arc $\Upsilon \subseteq X$ with end points x and $z\}$.*

Using Lemma 4.2, we may detect a great difference between non-degenerate and degenerate closed-Peano-components:

Lemma 4.7 *Every non-closed singleton is itself a closed-Peano- component.*

Lemma 4.8 *If every closed-Peano-component of the space X is non-degenerate, then X is T_1.*

Finally, closed-Peano-components act like connected components under Cartesian products.

Theorem 4.9 *Let Y be a closed-Peano-connected space and let X be an arbitrary space. Then X is T_1 and $\{E_k\}_{k\in K}$ are its closed- Peano-components if and only if $\{E_k \times Y\}_{k\in K}$ are the closed-Peano -components of $X \times Y$.*

Proof: Necessity: Let $\varphi : X \times Y \to X$ be the projection and let $k \in K$. Since X and Y are T_1 and any two points of E_k are closed-Peano-connected in X, we may deduce (reasoning as in Theorem 4.5, sufficiency), that every pair of points of $E_k \times Y$ is closed-Peano-connected in $X \times Y$. If $E_k \times Y$ is not a closed-Peano -component, then there exists a closed arc $\Upsilon \subseteq X \times Y$ such that $(E_k \times Y) \cap \Upsilon \neq \emptyset$ and $\Upsilon \not\subseteq E_k \times Y$. Hence $\varphi(\Upsilon) \subseteq X$ is a closed Peano continuum (by Lemma 3.3) and, moreover, $E_k = \varphi(E_k \times Y)$, $E_k \cap \varphi(\Upsilon) \neq \emptyset$ and $\varphi(\Upsilon) \not\subseteq E_k$, which is a contradiction.

Sufficiency: Given $k \in K$, since any two points of $E_k \times Y$ are closed-Peano-connected in $X \times Y$, we conclude (arguing as in Theorem 4.5, necessity), that every pair of points of E_k are closed-Peano-connected in X. If E_k is not a closed-Peano-component, there exists a closed arc $\Upsilon \subseteq X$ such that $E_k \cap \Upsilon \neq \emptyset$ and $\Upsilon \not\subseteq E_k$. Taking now a closed arc $T \subseteq Y(Y$ is non-degenerate), we deduce that $\Upsilon \times T \subseteq X \times Y$ is a closed Peano continuum, $(E_k \times Y) \cap (\Upsilon \times T) \neq \emptyset$ and $\Upsilon \times T \not\subseteq E_k \times Y$, a contradiction. On the other hand, since Y is non-degenerate, every closed-Peano-component of $X \times Y$ is non-degenerate and thus $X \times Y$ is T_1 (by Lemma 4.8). But $X \times \{y\} \subseteq X \times Y$ with $y \in Y$, so X is T_1 as well. ■

When working in non-Hausdorff spaces, the arcwise connectedness property is not invariant under continuous mappings and the relation of being arcwise connected is seldom an equivalence relation (see [4], [6] and [8]). However, if the arcs in definition 2.1.5 are required to be *closed* subsets of X (recall definition of closed-Peano-connectedness), then we obtain an equivalence relation (the relation of being closed-Peano-connected). Moreover, the closed-Peano-connectedness is invariant under continuous closed mappings as well. On the other hand, we can now relax the hypotheses in Example 3.5.

5 Applications of Closed-Peano-Connectedness

In this section we present some applications of the above theorems.

Theorem 5.1 *Let A, B, X and Y be spaces such that $A \times X \cong B \times Y$. If X and Y are closed-Peano-connected; A is T_1 and every closed-Peano-component of A(or B) is a singleton; then $A \cong B$ and $X \cong Y$.*

Proof: Note that $B \times Y$ is T_1 because so are A and X. But $B \times \{y\} \subseteq B \times Y$ with $y \in Y$, so B is T_1 as well. On the other hand, it is easy to prove that the three following propositions are equivalent in the T_1 space H:

(i) every closed-Peano-component of H is a singleton,
(ii) every closed Peano continua contained in H is a singleton,
(iii) H contains no closed arcs.

Indeed, (i) implies (ii) by definition, (ii) implies (iii) by Lemma 4.1, and finally, (iii) implies (i) by Lemma 4.6. Hence, the theorem conclusion follows then from Example 3.5. ■

The following example shows that the condition that A is T_1 cannot be removed from the hypotheses in Theorem 5.1.

Example 5.2 Let A be any countable space for which every singleton $\{x\} \subseteq A$ is not closed. Since $\mathcal{I}$ is compact, the projection $\varphi : A \times \mathcal{I} \to A$ is a closed mapping (see [4]), and no singleton contained in $A \times \mathcal{I}$ is closed. Then, from lemma 4.7, every closed-Peano-component of A is a singleton, and similarly for $A \times \mathcal{I}$. On the other hand, $\mathcal{I}^w$ is clearly closed-Peano- connected and $(A \times \mathcal{I}^w) \cong (A \times \mathcal{I} \times \mathcal{I}^w)$ but $A \ncong (A \times \mathcal{I})$. ■

Now, we dedicated the last part of this paper to improve Theorem 5.1 using the results of section 3, in particular Theorems 3.2 and 3.6.

Theorem 5.3 *Let X and Y be separable metric, pathwise connected spaces with dim $(X) \leq$ dim $(Y) < \infty$, and let A be a T_1 space containing no closed arcs. Then, the law $A \times X \cong B \times Y \Leftrightarrow A \cong B$ and $X \cong Y$ holds for any arbitrary space B.*

Proof: Let g be a homeomorphism from $A \times X$ onto $B \times Y$. Since X and Y are KC, X and Y are closed-Peano-connected. Moreover, proceeding as in Theorem 5.1, we deduce that B is T_1.

Now then, we claim that every closed-Peano-component of B is a singleton. Indeed, suppose, on the contrary, that $\mathcal{E} \subseteq B$ is a non-degenerate closed-Peano-component. Then there exists a closed arc Υ contained in B such that $\Upsilon \subseteq \mathcal{E}$. Furthermore, by Theorem 4.9, $\mathcal{E} \times Y$ is a closed-Peano-component of $B \times Y$. On the other hand, since A contains no closed arcs (by Theorem 4.9 and the proof of Theorem 5.1) $\{\{a\} \times X\}_{a \in A}$ are the closed-Peano-components of $A \times X$. Then there exists a point $a \in A$ such that $g(\{a\} \times X) = \mathcal{E} \times Y$. Hence dim $(X) =$ dim $(\mathcal{E} \times Y) \geq$ dim $(\Upsilon \times Y) = 1 +$ dim (Y) (see [9]), which is a contradiction because dim $(X) \leq$ dim $(Y) < \infty$. Thus every closed-Peano-component of B is a singleton and the result follows from Theorem 5.1. ■

We introduce a new definition.

Definition 5.4 *In a closed-Peano-connected space $H, z \in H$ is a* Peano-cut-point *if $H - \{z\}$ is not closed-Peano-connected.*

Theorem 5.5 *Let A, B, X and Y be spaces such that $A \times X \cong B \times Y$. Assume that X and Y are closed-Peano-connected, X has a Peano-cut-point and A is T_1 containing no closed arcs. Then $A \cong B$ and $X \cong Y$.*

Proof: Proceed as in Theorem 3.6, applying Theorems 4.9 and 5.1. ∎

We consider Theorems 5.3 and 5.5 to be the main applications of Theorem 3.2. We next present some examples deduced from Theorem 5.3. **Corollary 5.6.** Let A be a T_1 space containing no copies of $\mathbb{R}$ and let H be a separable metric, pathwise connected and finite dimensional space. Then, for any arbitrary space B, the *factorization law* $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds.

Corollary 5.6 *The Factorization law $A \times H \cong B \times H \Leftrightarrow A \cong B$ holds when H is a separable metric, pathwise connected and finite dimensional space; A is a T_1 space containing no copies of $\mathbb{R}$; and B is an arbitrary space.*

Proof: Since every arc contains a copy of $\mathbb{R}$, A contains no closed arcs, and the result follows then from Theorem 5.3. ∎

Using the Example 1.2 of this paper, we deduce the following.

Example 5.7 The space X contains no copies of $\mathbb{R}$.

$\Downarrow +T_1$ $\qquad\qquad\qquad$ $\Uparrow$

For every $A \subseteq X$ and every space B, the unique
Factorization law $A \times [0,1) \cong B \times [0,1) \Leftrightarrow A \cong B$ holds.

Proof: [$\Downarrow$]. In this case we assume X is T_1. Thus A is also a T_1 space containing no copies of $\mathbb{R}$ and the result follows from corollary 5.6.
[$\Uparrow$]. Suppose that X contains a copy of $\mathbb{R}$. Then we can find a set $A \subseteq X$ homeomorphic to $\mathbb{R}$. Hence $A \not\subseteq \mathcal{I}$ and $A \times [0,1) \cong \mathcal{I} \times [0,1)$ according to the Example 1.2, a contradiction. ∎

From Fox's example (see Example 1.2), we can similarly show the following.

Example 5.8 The space X is T_1 and contains no copies of $\mathbb{R}$.

$$\Downarrow$$

For every $A \subseteq X$ and every space B, the unique factorization law $A \times \mathcal{I} \subseteq B \times \mathcal{I} \Leftrightarrow A \subseteq B$ holds.

$$\Downarrow$$

X contains no copies of $\mathbb{R}^2$.

In a forthcoming paper [10] we show that the converse to the first implication is not valid. We conjecture, however, that the converse of the second implication is true when X is a metric space.

Eduardo Santillán
Department of Mathematics and Statistics
York University
4700 Keele Street
North York M3J 1P3, Ontario
Canada.

References

[1] E. Behrends and J. Pelant, *The cancellation law for compact Hausdorff spaces and vector-valued Banach-Stone theorems*, Arch. Math. **64** (1995), 341-343.

[2] K. Borsuk, *Sur la décomposition des polyèdres en produits cartésiens*, Fund. Math. **31** (1938), 137-148.

[3] K. Borsuk, *On the decomposition of manifold into products of curves and surfaces*, Fund. Math. **33** (1945), 273-298.

[4] J. Dugundji, *Topology*, Allyn and Bacon, 1966.

[5] R.H. Fox, *On a problem of S. Ulam concerning cartesian products*, Fund. Math. **34** (1947), 278-287.

[6] A. García Máynez y A. Tamariz Mascarúa, *Topología General*, Editorial Porrúa, 1988.

[7] K. Kuratowski, *Topology*, Volume II, Academic Press, 1968.

[8] J. Margalef Roig, E.O. Domínguez y J.L. Pinilla Fernando, *Topología*, Editorial Alhambra, 1982.

[9] K. Nagami, *Dimension Theory*, Academic Press, 1970.

[10] E. Santillán, *Cancellation laws in products with GO spaces*, In preparation.