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On Configuration Spaces of Orbits of Points and their Loop Space Homology *

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Abstract

Certain spaces which are analogues of configuration spaces are studied. In addition, their homology and loop space homology are studied. The Lie algebra of primitives in the loop space is a "twisted" extension of free Lie algebras. One fact is that the "twisting" is given by analogues of the infinitesimal braid relations plus a new relation.

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1 Introduction

The integral homology of $\Omega F(\mathbb{R}^{2n}, q)$ has been computed by E. Fadell, S. Husseini (unpublished), and subsequently by F. Cohen and S. Gitler [2], who showed that the Lie algebra of primitives is given by a non-trivial extension of free Lie algebras. Namely, there is an isomorphism from the module of primitives $PH_*(\Omega F(\mathbb{R}^n, q)) \cong L_1 \oplus \ldots \oplus L_q$ where $L_i = L[A_{i,1}, \ldots, A_{i,i-1}]$, is a free Lie algebra on $i-1$ generators, and the relations among the $A_{i,j}$ are of the form:

$$\begin{array}{rcl} [A_{i,j}, A_{k,i}] & = & [A_{k,i}, A_{k,j}] \\ [A_{i,j}, A_{k,j}] & = & [A_{k,j}, A_{k,i}] \end{array} \qquad \text{for} \quad 1 \leq j < i < k \leq q$$

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and $[A_{i,j}, A_{k,l}] = 0$ for distinct i, j, k, l, where [,] denotes the Lie-bracket. These are known as the "infinitesimal braid relations" or "Yang-Baxter relations" and they have appeared independently in several contexts in algebra and topology. In [7], T. Kohno interpreted them as integrability conditions of certain connections in a complex vector bundle. One can obtain the relations in a slightly different setting, by considering the Lie algebra $E^0_*(\pi)$ associated with the descending central series of a group π. In [4] Falk and Randell were interested in the case when π is the fundamental group of a fiber-type arrangement. In particular, they computed the additive structure of $E^0_*(P_k)$ where P_k is the pure braid group on k strands. Apart from the grading, this is known to be isomorphic (as a Lie algebra) to $PH_*\Omega F(\mathbb{R}^{2n}, k)$ for $n \geq 2$, where P stands for the primitives.

In this article, analogues $F_G(M, k)$ of configuration spaces are defined. Loosely speaking, these are spaces of ordered k-tuples of points in a space M which has a G-action and where the points lie in distinct orbits. In the case of $G = \mathbb{Z}/p$, $M = \mathbb{C} - \{0\}$ they also provide interesting examples of fiber-type arrangements.

2 The spaces $F_G(M, k)$

Let M be an n-dimensional manifold, G a finite group and let us assume that G acts freely on M. Let Gm denote the orbit of an element $m \in M$ under the action of G. Inspired by [5], define

$$F_G(M, k) = \{\, (m_1, \ldots, m_k) \in M^k \mid \; Gm_i \neq Gm_j \;\text{ for }\; i \neq j \,\}$$

For any natural number i, fix a finite subset $Q_i \subset M$ with cardinality $|Q_i| = i$. Then the spaces $F_G(M, k)$ satisfy the following:

Theorem 2.1.1 *For $k \geq l$, the projection $p : F_G(M, k) \to F_G(M, l)$ onto the first l coordinates, is a locally trivial bundle with fiber $F_G(M - Q_{|G|l}, k - l)$.*

An equivalent definition can be given in terms of ordinary configuration spaces. Let $f : M/G \to BG$ be the map that classifies the covering space $G \to M \to M/G$.

Theorem 2.1.2 *The space $F_G(M, k)$ is homeomorphic to the total space of the pull-back of the principal fibration $G^k \to (EG)^k \to (BG)^k$ along the composition $F(M/G, k) \hookrightarrow (M/G)^k \xrightarrow{f^k} (BG)^k$.*

Therefore, one has a G^k-principal bundle $F_G(M,k) \to F(M/G,k)$ which can be described in the obvious way. Theorems 2.1 and 2.2 will be proved in section 5. Some examples of manifolds with free group-actions are:

1. $\mathbb{R}^n - \{\vec{0}\}$ with a $\mathbb{Z}/2$-action given by the antipodal map.
2. $\mathbb{C}^n - \{\vec{0}\}$ with a a $\mathbb{Z}/p$-action given by multiplication by a primitive p-th root of unity ζ_p.
3. The actions in 1. and 2. restrict to free actions of $\mathbb{Z}/2$ and $\mathbb{Z}/p$ on the spheres S^n and S^{2n+1}, respectively.
4. For any manifold M, the symmetric group on k letters Σ_k acts freely on the configuration space $F(M,k)$.

Using a very well known result about fibrations with cross-sections (see section 3.2), one can prove:

Theorem 2.1.3 *If the fibration* $(M - Q_{|G|(i-1)}) \to F_G(M,i) \to F_G(M, i-1)$ *has a cross-section for* $2 \leq i \leq k$*, then there is a homotopy equivalence:*

$$\Omega F_G(M,k) \simeq \prod_{i=0}^{k-1} \Omega(M - Q_{|G|i})$$

Remark This is always the case when $M = M' - \{*\}$ is a manifold with a "puncture" or $M = N \times \mathbb{R}$. This product decomposition however is not multiplicative, i.e, *it is not a homotopy equivalence of H-spaces*. The precise algebraic extension in homology is given next for the space $B_k(\mathbb{R}^n)$ which is defined below.

Set $B_k(R^n) = F_{\mathbb{Z}/2}(\mathbb{R}^n - \{\vec{0}\}, k)$, with the $\mathbb{Z}/2$-action given in 1. The notation comes from the fact that $B_k(\mathbb{C}) \subset \mathbb{C}^k$ is the complement of an arrangement of hyperplanes of type B_k (see [10]). The arrangement in question is the union of the complexifications of the reflecting hyperplanes in a Coxeter group of type B_k. In the next section we study the homology of $B_k(\mathbb{R}^n)$ as well as the homology ring of its loop space. A sample of the homological calculation is given by the following theorem. For any non-zero integer j, define $\mathrm{sgn}(j) = j/|j|$.

Theorem 2.1.4 *For an even integer $n \geq 4$, the Lie algebra of primitives in the Hopf algebra $H_*(\Omega B_q(\mathbb{R}^n))$ is given as a graded module by:*

$$PH_*(\Omega B_q(\mathbb{R}^n)) \cong L[V_1] \oplus L[V_2] \oplus \ldots \oplus L[V_q],$$

where $L[V_i]$ is the free Lie algebra generated by the set $V_i = \{ B_{i,j} \mid i > |j| \}$ (to be defined in section 3). The Lie products $[a,b]$ with $a \in V_i$ and $b \in V_k$ for $1 \leq i < k \leq q$ are given as follows:

If $\{i, |j|\} \cap \{k, |l|\} = \emptyset$, then $[B_{i,j}, B_{k,l}] = 0$. Otherwise,

(a) *if* $j \neq 0 \neq l$, *then*

$$[B_{i,j}, B_{k,l},] = \begin{cases} [B_{k,l}, B_{k,\mathrm{sgn}(l)j}] & \text{for } |l| = i \\ \\ [B_{k,l}, B_{k,\mathrm{sgn}(l)\mathrm{sgn}(j)i}] & \text{for } |l| = |j|; \end{cases}$$

(b) *if* $j = 0$, *then*

$$[B_{i,0}, B_{k,l}] = [B_{k,l}, (B_{k,-i} + B_{k,0} + B_{k,i})].$$

3 The homology ring $H_*(\Omega B_q(\mathbb{R}^n))$

3.1 The homology of $B_q(\mathbb{R}^n)$

In this section we compute $H_* B_q(\mathbb{R}^n)$ and construct an explicit basis for $H_{n-1}(B_q(\mathbb{R}^n))$. We already know that the fibration: $(\mathbb{R}^n - Q_{2q-1}) \to B_q(\mathbb{R}^n) \to B_{q-1}(\mathbb{R}^n)$ has a section; therefore $H_* B_{q-1}(\mathbb{R}^n)$ injects in the homology of the total space and the corresponding Serre spectral sequence collapses for dimensional reasons, so

$$\begin{aligned} H_*(B_q(\mathbb{R}^n)) &\cong H_*(B_{q-1}(\mathbb{R}^n)) \otimes H_*(\mathbb{R}^n - Q_{2q-1}) \\ &\cong H_*(B_{q-1}(\mathbb{R}^n)) \otimes H_*(\bigvee_{2q-1} S^{n-1}) \\ &\cong \bigotimes_{i=1}^{q} H_*(\bigvee_{2i-1} S^{n-1}) \qquad \text{(by induction)}, \end{aligned}$$

and, in particular, H_{n-1} has rank $1+3+\ldots+(2q-1) = q^2$. Let $\vec{e}_1 \in \mathbb{R}^n$ be the first canonical unit vector. Now, for $i = 0, \ldots, q$ put $x_i = i \cdot \vec{e}_1$

For $0 \leq j < i < q$ let

$$C_{i,j}, \;\; \bar{C}_{i,j} : S^{n-1} \longrightarrow B_q(\mathbb{R}^n)$$

be given by:

$$C_{i,j}(z) = (x_1, x_2, \ldots, x_{i-1}, x_j + \frac{z}{2}, x_{i+1}, \ldots, x_q) \qquad \text{if } j \geq 0$$

$$\bar{C}_{i,j}(z) = (x_1, x_2, \ldots, x_{i-1}, -x_j + \frac{z}{2}, x_{i+1}, \ldots, x_q) \qquad \text{if } j > 0.$$

Let ι be the fundamental class in $H_*(S^{n-1})$. Then the set of homology classes

$$\{\, C_{i,j*}(\iota) \mid 0 \leq j < i < q \,\} \cup \{\bar{C}_{i,j*}(\iota) \mid 1 \leq j < i < q \,\}$$

is the desired basis.

We can prove the $C_{i,j}$'s are linearly independent by constructing their duals in cohomology. For $0 \leq j < i < q$ let $p_{i,j},\ p^+_{i,j} : B_q(\mathbb{R}^n) \longrightarrow S^{n-1}$ be the maps

$$p_{i,j}(\vec{x}) = \frac{x_i - x_j}{|x_i - x_j|} \qquad \text{if } j \geq 0,$$

$$p^+_{i,j}(\vec{x}) = \frac{x_i + x_j}{|x_i + x_j|} \qquad \text{if } j > 0.$$

The induced maps

$$p_{i,j*}, \;\; p^+_{i,j*} : H_{n-1}(B_q(\mathbb{R}^n)) \longrightarrow \mathbb{Z}$$

represent elements in $H^{n-1}(B_q(\mathbb{R}^n))$ dual to $C_{i,j*}(\iota)$ and $\bar{C}_{i,j*}(\iota)$. This can be seen by evaluating directly $< p^+_{i,j*}, \mathcal{C} >$, where $\mathcal{C}$ runs through the set constructed above. Finally, if we choose $(x_1, \ldots, x_k)$ to be the base point of $B_k(\mathbb{R}^n)$, we can identify the set

$$\{C_{k,0*}(\iota)\} \cup \{\, C_{k,j*}(\iota) \mid 1 \leq j < k \,\} \cup \{\, \bar{C}_{k,j*}(\iota) \mid 1 \leq j < k \,\}$$

with a basis for $H_{n-1}(\mathbb{R}^n - Q_{2k-1}) \cong H_{n-1}(\vee_{2k-1} S^{n-1})$.

3.2 The Lie algebra of primitives in $H_*(\Omega B_q(\mathbb{R}^n))$

We recall the following fact (see for example [1]). Let $F \to E \to B$ be a fibration with a cross-section $\sigma : B \to E$. Then there is a homotopy equivalence: $\Omega B \times \Omega F \longrightarrow \Omega E$. A choice of equivalence is

$$\Omega B \times \Omega F \xrightarrow{\Omega(\sigma)\times\Omega(\text{incl})} \Omega E \times \Omega E \xrightarrow{\text{mult}} \Omega E$$

Therefore

(1) The inclusions $\Omega B \to \Omega E$ and $\Omega F \to \Omega E$ are multiplicative and thus, the induced maps in homology are morphims of algebras.

(2) $H_*(\Omega B) \otimes H_*(\Omega F) \longrightarrow H_*(\Omega E)$ is an isomorphism if coefficients are taken such that the strong form of the Künneth theorem holds (where Tor $= 0$).

This implies Theorem 2.1.3 and gives the module structure of

$$H_*(\Omega B_q(\mathbb{R}^n)).$$

Indeed, (2) can be applied to the tower of fibrations

$$\begin{array}{ccc}
B_q(\mathbb{R}^n) & \longleftarrow & (\mathbb{R}^n - Q_{2q-1}) \\
\downarrow & & \\
B_{q-1}(\mathbb{R}^n) & \longleftarrow & (\mathbb{R}^n - Q_{2q-3}) \\
\downarrow & & \\
\vdots & & \vdots \\
\downarrow & & \\
B_2(\mathbb{R}^n) & \longleftarrow & (\mathbb{R}^n - Q_3) \\
\downarrow & & \\
B_1(\mathbb{R}^n) & \longleftarrow & (\mathbb{R}^n - \{0\})
\end{array}$$

to set up an isomorphism

$$\bigotimes_{i=1}^{q} H_*\, \Omega(\bigvee_{2i-1} S^{n-1}) \longrightarrow H_*(\Omega B_q(\mathbb{R}^n)).$$

By the Bott-Samelson Theorem (see [1]),

$$H_*\Omega(\bigvee_{2i-1} S^{n-1}) \cong T[\bar{H}_*(\bigvee_{2i-1} S^{n-2})]$$

and then, as graded modules, there is an isomorphism

$$H_*(\Omega B_q(\mathbb{R}^n)) \cong T[V_1] \otimes T[V_2] \otimes \ldots \otimes T[V_q]$$

where $T[V]$ denotes the tensor algebra generated by the set V and V_i is a basis for $H_{n-2}(\vee_{2i-1} S^{n-2}) \cong H_{n-1}(\vee_{2i-1} S^{n-1})$ (the isomorphism is given by the homology suspension). Thus, to calculate the ring structure of $H_*(\Omega B_q(\mathbb{R}^n))$ it suffices to compute the commutators $[a,b]$ with $a \in V_i$ and $b \in V_k$ for $1 \leq i < k \leq q$. The following is a very well known result:

Lemma 3.2.1 *Let X be 1-connected and assume that $H_*(\Omega X;\mathbb{Z})$ is primitively generated (as a Hopf algebra) and torsion-free. Then*

$$H_*(\Omega X;\mathbb{Z}) \cong U(P(H_*(\Omega X;\mathbb{Z})))$$

is the universal enveloping algebra of the Lie algebra of primitives.

Proof: By hypothesis, the natural homomorphism $U(PH_*(\Omega X;\mathbb{Z})) \to H_*(\Omega X;\mathbb{Z})$ is a surjection. Notice that after tensoring with $\mathbb{Q}$ it also becomes an injection (see for example [9]). Since there is no torsion this implies the original map was an injection. ■

Now, the module of primitives in $H_*\Omega(\vee_{2i-1} S^{n-1};\mathbb{Z})$ is given by $L[V_i]$, the free Lie algebra generated by V_i. Finally, the module of primitives in $H_*X \otimes H_*Y$ (for path connected X and Y) is $PH_*X \oplus PH_*Y$. Thus there is an isomorphism

$$PH_*(\Omega B_k(\mathbb{R}^n)) \cong L[V_1] \oplus L[V_2] \oplus \ldots \oplus L[V_q]$$

as graded modules and, for every i, $L[V_i]$ is a Lie subalgebra of

$$PH_*(\Omega B_q(\mathbb{R}^n)).$$

The problem is now reduced to determining the Lie algebra structure on $PH_*(\Omega B_q(\mathbb{R}^n))$.

Consider now the maps induced in homology by the loops of the $C_{i,j}$'s :

$$(\Omega C_{i,j})_*, (\Omega \bar{C}_{i,j})_* : H_*(\Omega S^{n-1}) \longrightarrow H_*(\Omega B_q(\mathbb{R}^n))$$

and use them to define

$$\begin{aligned} B_{i,j} &= (\Omega C_{i,j})_*(\iota_{n-2}) \\ \bar{B}_{i,j} &= (\Omega \bar{C}_{i,j})_*(\iota_{n-2}). \end{aligned}$$

We take now V_i to be the set $\{ B_{k,0}\} \cup \{B_{k,j} \mid 1 \leq j < k \} \cup \{ \bar{B}_{k,j} \mid 1 \leq j < k \}$

Let α denote the integer $(-1)^n$.

Theorem 3.2.2 *The Lie algebra of primitives in the Hopf Algebra*

$$H_*(\Omega B_q(\mathbb{R}^n))$$

is given as a graded module by:

$$PH_*(\Omega B_q(\mathbb{R}^n)) = L[V_1] \oplus L[V_2] \oplus \ldots \oplus L[V_q]$$

The Lie products $[a, b]$ with $a \in V_i$ and $b \in V_k$ for $1 \leq i < k \leq q$ are given as follows:

If $\{i, j\} \cap \{k, l\} = \emptyset$, then

$$[B_{i,j}, B_{k,l}] = [B_{i,j}, \bar{B}_{k,l}] = [\bar{B}_{i,j}, B_{k,l}] = [\bar{B}_{i,j}, \bar{B}_{k,l}] = 0$$

Otherwise, if

(a) $j = l$ *then* $\begin{cases} [B_{i,j}, B_{k,j}] = [B_{k,j}, B_{k,i}] & (j \neq 0) \\ [B_{i,0}, B_{k,0}] = [B_{k,0}, B_{k,i},] + \alpha[B_{k,0}, \bar{B}_{k,i}] & \end{cases}$

(b) $i = l$ *then* $\begin{cases} [B_{i,j}, B_{k,i}] = \alpha[B_{k,i}, B_{k,j}] & (j \neq 0) \\ [B_{i,0}, B_{k,i}] = \alpha[B_{k,i}, B_{k,0}] + [B_{k,i}, \bar{B}_{k,i}] & \end{cases}$

(c) $j = l$ *then* $[B_{i,j}, \bar{B}_{k,j}] = \alpha[\bar{B}_{k,j}, \bar{B}_{k,i}]$ $(j \neq 0)$

(d) $i = l$ *then* $\begin{cases} [B_{i,j}, \bar{B}_{k,i}] = [\bar{B}_{k,i}, \bar{B}_{k,j}] & (j \neq 0) \\ [B_{i,0}, \bar{B}_{k,i}] = [\bar{B}_{k,i}, B_{k,0}] + [\bar{B}_{k,i}, B_{k,i}] & \end{cases}$

(e) $j = l$ *then* $[\bar{B}_{i,j}, B_{k,j}] = \alpha[B_{k,j}, \bar{B}_{k,i}]$

(f) $i = l$ *then* $[\bar{B}_{i,j}, B_{k,i}] = \alpha[B_{k,i}, \bar{B}_{k,j}]$ $(j \neq 0)$

(g) $j = l$ *then* $[\bar{B}_{i,j}, \bar{B}_{k,j}] = [\bar{B}_{k,j}, B_{k,i}]$

(h) $i = l$ *then* $[\bar{B}_{i,j}, \bar{B}_{k,i}] = [\bar{B}_{k,i}, B_{k,j}]$ $(j \neq 0)$.

In the case of even n, most of the relations are redundant and the statement simplifies considerably if we introduce a change of notation. For $|j| < i$, let

$$B_{i,j} = \begin{cases} B_{i,j} & \text{if } j \geq 0 \\ \bar{B}_{i,|j|} & \text{if } j < 0. \end{cases}$$

Then we have:

Theorem 3.2.3 *For even $n \geq 4$, the Lie algebra of primitives in the Hopf Algebra $H_*(\Omega B_q(\mathbb{R}^n))$ is given as a graded module by:*

$$PH_*(\Omega B_q(\mathbb{R}^n)) = L[V_1] \oplus L[V_2] \oplus \ldots \oplus L[V_q].$$

The Lie products $[a, b]$ with $a \in V_i$ and $b \in V_k$ for $1 \leq i < k \leq q$ are :

If $\{i, |j|\} \cap \{k, |l|\} = \emptyset$, then $[B_{i,j}, B_{k,l}] = 0$

Otherwise,

(a) *if* $j \neq 0 \neq l$ *then*

$$[B_{i,j}, B_{k,l}] = \begin{cases} [B_{k,l}, B_{k,\mathrm{sgn}(l)j}] & \textit{if } |l| = i \\ [B_{k,l}, B_{k,\mathrm{sgn}(l)\mathrm{sgn}(j)i}] & \textit{if } |l| = |j|. \end{cases}$$

(b) *if* $j = 0$ *(and therefore $l = -i, 0, i$)*

$$[B_{i,0}, B_{k,l}] = [B_{k,l}, (B_{k,-i} + B_{k,0} + B_{k,i})]$$

Summarizing, the Lie brackets among primitive elements are given in terms of the "infinitesimal braid relations" with signs (a), and a "new relation" (b) and they provide the set of commutation rules in $H_*(\Omega B_q(\mathbb{R}^n))$. We know that the presence of the signs is due to the $\mathbb{Z}/2$ action considered, since analogous calculations have been carried out for the $\mathbb{Z}/p$ case, with the generators of the form $B^g_{i,j}$, $g \in \mathbb{Z}/p$ (see [12]). The new relations appeared naturally since the manifold M considered here was not contractible.

3.3 Lie relations

To compute the Lie brackets $[B_{i,j}, B_{k,l}]$, with $|j| < i$, $|l| < k$ for $1 \leq i < k \leq q$ we proceed as follows. By induction, it suffices to do the case when $k = q$. We construct the specific map

$$\varphi : S^{n-1} \times S^{n-1} \longrightarrow B_q(\mathbb{R}^n)$$

given by $\varphi(z, w) = (y_1, y_2, \ldots, y_q)$, where

$$y_s = \begin{cases} x_s & \text{if } s \neq i, q \\ \mathrm{sgn}(j) \cdot x_j + z/2 & \text{if } s = i \\ \mathrm{sgn}(l) \cdot y_i + w/4 & \text{if } s = q \end{cases}$$

and compute the images of the fundamental cycles in $H_*(S^{n-1} \times S^{n-1})$, with the aid of the dual classes $p_{i,j}$ and $p^+_{i,j}$

$$\begin{aligned} \varphi_*(\iota_{n-1} \otimes 1) &= \sum C_{r,s*}(\iota) \\ \varphi_*(1 \otimes \iota_{n-1}) &= \sum C_{t,u*}(\iota). \end{aligned}$$

We know that after looping, the cycles $(\iota_{n-2} \otimes 1)$ and $(1 \otimes \iota_{n-2})$ commute (in the graded sense) in $H_*(\Omega(S^{n-1} \times S^{n-1}))$ and it follows by naturality that $[\sum B_{r,s}, \sum B_{t,u}] = 0$. This method provides us with all the relations. See [12] for the details.

4 Relation to descending central series.

There is a further relation to the Lie algebra associated to the descending central series of the braid groups.

Recall that given a group G, its descending central series $G = \Gamma^1 \supset \Gamma^2 \supset \ldots$ is defined as: $\Gamma^1 = G$, $\Gamma^n = [\Gamma^{n-1}, G]$. Notice that Γ^{n+1} is normal in Γ^n and so we can form the associated graded group of this filtration $\{E^0_n\}_{n \geq 1}$ by setting $E^0_n(G) = \Gamma^n/\Gamma^{n+1}$, which is an abelian group for all n. Put $E^0_*(G) = \bigoplus_n E^0_n(G)$. Now for every p and q the commutators $[a, b] = aba^{-1}b^{-1}$ in G induce a bilinear map

$$E^0_p \times E^0_q \longrightarrow E^0_{p+q}$$

which makes $E^0_*(G)$ into a Lie algebra. Moreover, E^0_* is a functor from the category of groups to the category of Lie algebras. Two elementary properties of E^0_* are:

1. $E_1^0(G) = H_1(G;\mathbb{Z})$ (with $\mathbb{Z}$-trivial coefficients).
2. If $F[X]$ is the free group generated by a set X, then $E_*^0(F[X])$ is isomorphic to $L[X]$, the free Lie algebra generated by X (see [11]).

In [4], Falk and Randell have proved:

Theorem 4.1.1 *Let* $1 \to A \overset{i}{\to} B \overset{j}{\to} C \to 1$ *be a split extension of groups such that* C *acts trivially on* $H_1(A)$*. Then, the induced sequence*

$$0 \to E_*^0(A) \to E_*^0(B) \to E_*^0(C) \to 0$$

is split exact (as graded abelian groups, but not necessarily as Lie algebras).

Sketch of the proof. Let $\sigma : C \to B$ be a cross-section, i.e. a right-inverse of j. There is a well defined map $\tau : B \to A$ given by $\tau(b) = i^{-1}[\sigma j(b^{-1}) \cdot b]$ which need not be a homomorphism, but still is a left-inverse of i. Now, the triviality of the action is equivalent to $[A, C] \subset [A, A] = \Gamma^2 A$, where we identify A and C with their images under i and σ. One can prove inductively that $[\Gamma^n A, \Gamma^k C] \subset \Gamma^{n+k} A$ and use this to show that $\tau(\Gamma^n B) \subset \Gamma^n A$. Now one can show that the induced sequence

$$1 \to \Gamma^n A \to \Gamma^n B \to \Gamma^n C \to 1$$

is (split) exact, since $b \in \Gamma^n B \cap \ker j$ implies $b = i(\tau(b))$. The theorem follows easily from this last statement. ∎

Thus, there is an isomorphism $E_*^0(B) \cong E_*^0(A) \oplus E_*^0(C)$ of abelian groups. In general, this is not a trivial extension of Lie algebras but under the same hypothesis we can prove that if $a \in E_*^0(A)$ and $c \in E_*^0(C)$ then $[a, c] \in E_*^0(A)$.

Recall that the configuration space $F(\mathbb{R}^2, k)$ is a $K(\pi, 1)$ and its fundamental group can be identified with the pure braid group on k strands, P_k. From [3] and [5] we know that there is a fibration:

$$(\mathbb{R}^2 - Q_{k-1}) \longrightarrow F(\mathbb{R}^2, k) \longrightarrow F(\mathbb{R}^2, k-1)$$

which has a cross-section and trivial local coefficients in homology. Thus by looking at its exact sequence of homotopy groups, we obtain a split extension of fundamental groups

$$1 \longrightarrow F_{k-1} \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow 1$$

which satisfies the hypothesis of the theorem. Here F_{k-1} is a free group on $(k-1)$ generators, usually denoted $A_{k,1}, A_{k,2}, \dots, A_{k,k-1}$. Applying inductively Theorem 4.1 we get

$$\begin{aligned} E^0_*(P_k) &\cong E^0_*(F_1) \oplus E^0_*(F_2) \oplus \dots \oplus E^0_*(F_k) \\ &\cong L_1 \oplus L_2 \oplus \dots \oplus L_k \end{aligned}$$

with $L_i = L[A_{i,1}, \dots, A_{i,i-1}]$. We can work out the Lie algebra extension by using a presentation of P_k and obtain the infinitesimal braid relations among the $A_{i,j}$'s.

5 Proof of theorems in section 2

Lemma 5.1.1 *Let $V = (D^n)^\circ \subset \mathbb{R}^n$ be the interior of the n-dimensional unit disc. Then for every $x_0 \in V$ there is a homeomorphism $\phi_{x_0} : \bar{V} \to \bar{V}$ such that*

(a) $\phi|_{\partial V} = id_{\partial V}$ *(fixes the boundary).*

(b) $\phi_{x_0}(x_0) = 0$.

(c) *The map $\psi : V \times \bar{V} \to \bar{V}$ defined by $\psi(x, y) = \phi_x(y)$, is continuous.*

Proof: For $v \in \mathbb{R}^n$, let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ be the translation by $-v$, $T_v(x) = x - v$. Let $g : V \to \mathbb{R}^n$ be the homeomorphism: $g(x) = \frac{x}{1-|x|}$. Now, the composition

$$\begin{array}{ccc} V & \overset{\tilde{\phi}}{\dashrightarrow} & V \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle g^{-1}} \\ \mathbb{R}^n & \xrightarrow{T_{g(x_o)}} & \mathbb{R}^n \end{array}$$

gives a homeomorphism of V and can be extended continuously to $\phi : \bar{V} \to \bar{V}$ by defining $\phi(x) = x, \ \forall x \in \partial V$. Thus ϕ is the desired map.

It is now easy to express the restriction $\psi|_{V \times V}$ as the composite of continuous maps, and we can extend it to $V \times \bar{V}$ by $\psi(x, y) = y \ \ \forall x \in \partial V$. ∎

Proof (of Theorem 2.1.1). We adapt the proof in [5] to prove the case $l = k - 1$. The general case is similar. Fix a base point $(\xi_1, \ldots, \xi_{k-1}) \in F_G(M, k-1)$. Consider Euclidean neighborhoods $\xi_i \in V_i \subset M$ such that

(1) $V_i \cap V_j = \emptyset \quad i \neq j$

(2) $V_i \cap gV_j = \emptyset \quad \forall i, j$ and $g \neq e$

where $gV_j = \{ gx \mid x \in V_j \}$. Notice now that $V = V_1 \times \ldots \times V_{k-1}$ is a neighborhood about $(\xi_1, \ldots, \xi_{k-1})$. Using the previous lemma, construct maps $\theta_i : V_i \times \bar{V}_i \to \bar{V}_i$ satisfying:

(i) $\theta_i(x, -) : \bar{V}_i \to \bar{V}_i$ is a homeomorphism fixing ∂V_i

(ii) $\theta_i(x, x) = \xi_i$

(in some sense, $\theta_i|_x : \bar{V}_i \to \bar{V}_i$ are homeomorphisms parametrized by x), and use them to define the map $\theta : V \times M \to M$,

$$\theta(x_1, \ldots, x_l, y) = \begin{cases} y & \text{if } y \notin \bigcup_i \bigcup_g gV_i \\ g\theta_i(x_i, g^{-1}y) & \text{if } y \in gV_i \end{cases}$$

By construction, θ has the following property: $y \notin \{ gx_i \mid i = 1, \ldots, k-1;\ g \in G \}$ if and only if $\theta(\vec{x}, y) \notin \{ g\xi_i \mid i = 1, \ldots, k-1;\ g \in G \}$. Now, the local trivialization

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow[\approx]{\phi} & V \times (M - Q_{|G|l}) \\ \downarrow p & & \downarrow pr_1 \\ V & \xrightarrow{=} & V \end{array}$$

is given by: $\phi(\vec{x}, y) = (\vec{x}, \theta(\vec{x}, y))$ with inverse $\phi^{-1}(\vec{x}, z) = (\vec{x}, \theta^{-1}(\vec{x}, z))$. ■

Proof (of Theorem 2.1.2). Notice that the group $G^k = G \times \ldots \times G$ acts coordinate-wise on the product M^k. This induces a free action of G^k on $F_G(M, k)$ and its orbit space $F_G(M, k)/G^k$ can be identified with $F(M/G, k)$. Now, it follows from the definition of f that the covering space $G^k \to M^k \to (M/G)^k$ is classified by the map $f^k : (M/G)^k \to (BG)^k$ and thus we have

$$F_G(M,k) = i^*(M^k) = i^*(f^k)^*(EG)^k$$

where $i : F(M/G,k) \hookrightarrow (M/G)^k$ is the natural inclusion. ■

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